Representations of electric and magnetic fields

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Abstract

We propose for the graphical representation of fields, not to limit to field lines but also to draw the orthogonal surfaces. Just as the ends of the electric or magnetic field lines tell us where the sources of the flux of a field are located, the borders of the orthogonal surfaces indicate us the sources of the field's circulation.

I. Introduction

It is common practice to represent fields graphically. On a field diagram one often sees at a single glance what otherwise would take a greater effort to recognize. Even a qualitative picture can be an important teaching aid.

There are several possibilities of realizing a picture of the invisible system "field" [1]. First, one has the choice between the various physical quantities which might be represented. In the case of the electromagnetic field we can choose between the field vector quantities $\mathbf{E}(x,y,z)$, $\mathbf{D}(x,y,z)$, $\mathbf{H}(x,y,z)$ and $\mathbf{B}(x,y,z)$, the electric and the magnetic scalar potentials, the magnetic vector potential, the distribution of $\mathrm{div}\,\mathbf{D}(x,y,z)$ and $\mathrm{div}\,\mathbf{B}(x,y,z)$, the distribution of $\mathrm{curl}\,\mathbf{E}(x,y,z)$ and $\mathrm{curl}\,\mathbf{H}(x,y,z)$, the energy density, the energy flow density (which is similar to the momentum density) and the momentum flow.

Next, one can choose between several ways of representing the selected quantity. Scalar fields like the scalar potentials, the energy density or the divergence can be illustrated by contour lines, or by a color code. Vector fields like the field strengths, the curl of the field or the energy flow can be visualized by arrows placed at the vertices of a square grid or by field lines or stream lines, respectively. Tensor fields like the momentum flow density can be represented by three field line pictures [2], or in some cases by a field of ellipsoids. Although the various pictures of one and the same field are not independent of one another, each of them has its specific advantages.

Among the various representation modes of fields there is one which is by far more popular than the others: the representation of the field strengths by means of a field line diagram.

Field line diagrams have considerable advantages:

– With a single glance one gets an idea about the direction of the field vector in every point.

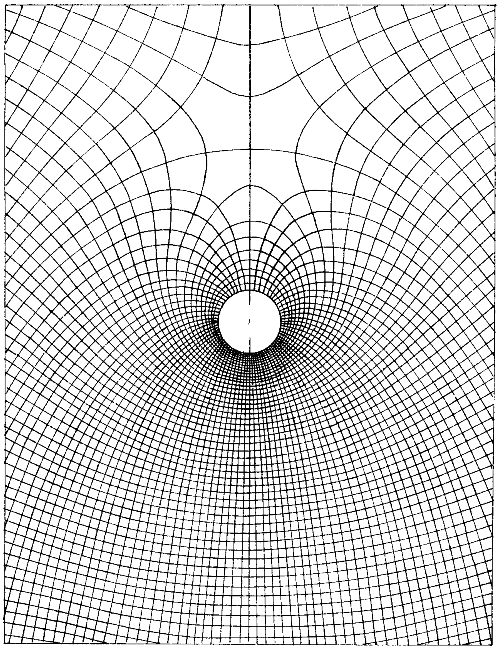

Fig. 1. Diagram of the superposition of the magnetic field of a linear current and a homogeneous magnetic field, in “Electricity and Magnetism”, by Maxwell. All of Maxwell's figures display “lines of force” and “level surfaces”.

– They clearly indicate where div **E**(x,y,z), div **B**(x,y,z) etc are non-zero. Thus they tell us the distribution of what we shall call the “flux sources” of the field. Therefore, they are a very immediate expression of the third and the fourth Maxwell equations, i. e. the fact that the flux sources of an electric field are located at the electric charges and that the magnetic **B** field has no such sources, respectively.

In the present article we argue that field line pictures become substantially more useful when completed by the corresponding orthogonal surfaces. Our proposal is to represent these surfaces not only in the case of electrostatic or magnetostatic potential fields, but also for electric and magnetic fields with non-zero circulation. Thus, in a two-dimensional representation, an electric or magnetic field appears as a grid of field lines and their orthogonal trajectories. Maxwell already used in all his graphical representations of fields – with and without circulation – “lines of force” and “level surfaces”, Fig. 1 [3]. The additional effort for this extension is not high. If colors are available, it is appropriate to represent the field lines in one color and the orthogonal surfaces in another.

Since our arguments are equally valid for electric and magnetic fields, for the sake of simplicity we shall mostly refer in the following to electric fields. Moreover, we limit our considerations to fields in the vacuum. Therefore, the shape of the **B** and **H** lines, as well as the shape of the **D** and **E** lines are the same.

As has already been mentioned, in a conventional electric field line plot one easily recognizes the distribution of div **E**, i. e. of the “flux sources” of the field. The distribution of curl **E**, on the contrary, can be recognized only with difficulty. When drawing the orthogonal surfaces –in addition to the field lines–, the “circulation sources” of the field, i. e. the places where curl **E** is non-zero, are just as clearly visible as the flux sources. In section III we shall discuss this subject by means of an example. An application which is particularly interesting is treated in section IV: the “rectangular electric field”.

In section II we shall begin with the discussion of a serious objection to the value of field line diagrams which was recently published in the American Journal of Physics [4].

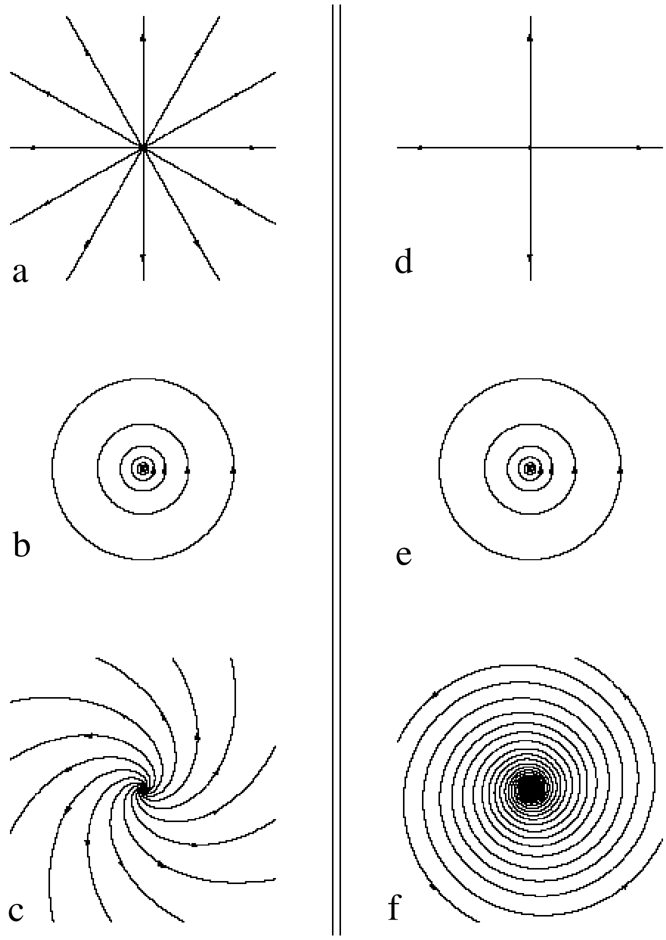

Fig. 2. (a) Electric field of a line charge. (b) Electric field of a thin solenoid with a magnetic flux changing at a constant rate. (c) Superposition of the fields of (a) and (b). (d) Field of a line charge with smaller charge density. (e) the same as (b). (f) Superposition of the fields of (d) and (e). The field line density suggests a higher magnitude of the field strength than the field actually has.

II. Do field line diagrams not work at all?

It is common practice to suggest or explicitly claim that the line density of field lines is a measure for the magnitude of the local field strength. Problems concerning this claim have been discussed recently in some articles in the American Journal of Physics [4, 5]. Wolf, Hook and Weeks [4] show that, for several independent reasons, the field line density does not represent the correct magnitude of the field strength. “Electric field line diagrams don't work”, states the title of their article.

Wolf et al limit their analysis to electrostatic fields. We want to back-up their argument by displaying one more “mechanism” which entails that the field line density does not correspond to the field strength. It can be observed whenever at any place of a field we have

$$\text{div}\,\mathbf{E} \neq 0$$

and at the same place or another place of the same field we have

$$\text{curl}\,\mathbf{E} \neq 0.$$

A very simple realization of this case is shown in Fig. 2. Fig. 2a displays the electric field of a line charge which is perpendicular to the drawing plane. Here, the field line density is proportional to the magni-

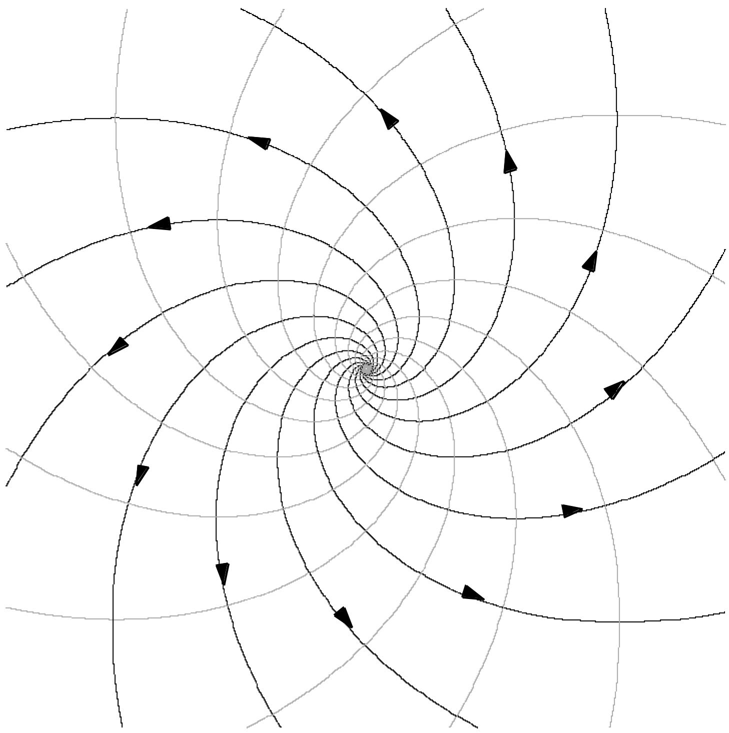

Fig. 3. The same field as that of Fig. 2c. In addition to the field lines the orthogonal lines are represented. Both ensembles of lines end within the small circle in the center of the figure. Thus, this circle must contain the flux sources as well as the circulation sources of the field.

tude of the field strength. Fig. 2b shows another simple situation: a thin solenoid of infinite extension perpendicular to the drawing plane. The electric current in the wire of the solenoid, and thus the magnetic flux within the solenoid is supposed to increase linearly with time. As a consequence, the solenoid is surrounded by an electric field which has circular field lines and which is constant in time. The field is similar to the magnetic field of an infinitely long current-carrying wire. Again the field lines can be drawn in such a way that their density is proportional to the magnitude of the field strength. Fig. 2c displays a superposition of the systems of figures 2a and 2b. Now, in the center of the figure we have both, a line charge and a magnetic flux which varies at a constant rate. The resulting field lines have the form of spirals. For the moment, this field line diagram seems not to be suspicious in any way. Let us now introduce a slight modification.

We replace the line charge of Fig. 2a by another one which is much weaker. Its field line diagram is Fig. 2d. We superpose the corresponding field with that of Fig. 2e, which is the same as that of Fig. 2b. The result is shown in Fig. 2f. The four field lines starting from the line charge at the center are spiraling very tightly around the central “source”. The smaller we choose the line charge the denser the field lines. By diminishing the line charge, the line density can be made as high as one desires. Apparently, in no place of Fig. 2f the field line density represents the field strength – even not approximately.

The reason for this incongruity is easily seen. In order to determine the field line density we choose a small reference surface perpendicular to the field lines. Now, the same field line passes several times through this surface and makes believe that many different field lines are crossing the area.

Two alternative conclusions could be drawn from these statements.

Wolf et al suggest to refrain from continuous line representations of fields altogether. In our view this would mean to empty the baby with the bath. Our proposal is: Take advantage of the virtues of conventional field line diagrams but don't pretend they represent something what they do not represent. In other words: Don't claim that the line density measures the field strength.

III. The borders of the orthogonal surfaces

A vector field $\mathbf{E}(x, y, z)$, vanishing sufficiently rapidly at spacial infinity, is completely specified if the scalar field div $\mathbf{E}(x, y, z)$ and the vector field curl $\mathbf{E}(x, y, z)$ are given [6]. If the field $\mathbf{E}(x, y, z)$ is plotted in a convenient manner, div $\mathbf{E}(x, y, z)$ and curl $\mathbf{E}(x, y, z)$ can be read directly from the diagram. However, it depends on the type of representation one has chosen, how easy this task is. If the representation is by field lines, it is most easy to recognize div $\mathbf{E}(x, y, z)$, i. e. the field's flux sources: The flux sources are located where field lines end. This is one of the basic rules we apply when drawing qualitative electric field line pictures, since in a typical problem the electric charges, i. e. the sources are given.

The circulation sources of the field, i. e. the function curl $\mathbf{E}(x, y, z)$, can be recognized in a field line picture only vaguely. Often, a field with curl $\mathbf{E} \neq 0$ can be distiguished from a field with curl $\mathbf{E} = 0$ only on thorough inspection. The fact that field lines are closed is rather eye-catching, but it only tells us that there are circulation sources somewhere inside the closed lines. It does not tell us exactly where they are. Moreover, there are fields which have circulation sources but no closed field lines at all. An example is the field of Fig. 2c. Another example will be considered in section IV.

The reason why it is so much easier to recognize the flux sources than the circulation sources is in the arbitrariness of our plot. Among the two mutually orthogonal series of lines, we have chosen the field lines and not the orthogonal trajectories. If we had chosen the orthogonal lines, we would have been able to read the circulation sources more easily than the flux sources. Indeed, the circulation sources are located at those places where the orthogonal surfaces end.

Before we show, why this is so, let us formulate our main conclusion: *When making a sketch of a field, draw both field lines and orthogonal surfaces* [7].

In a potential field – an electrostatic or a magnetostatic field – the orthogonal surfaces are called equipotential surfaces. An equipotential surface is a closed surface which is perpendicular to the field strength vector in every point. In a two-dimensional plot, the equipotential surfaces appear as closed lines.

As a matter of fact, orthogonal surfaces, or in two dimensions, orthogonal lines, can also be drawn when the field is not a potential field. The construction rule is the same as in the case of the potential field: the orthogonal surfaces are perpendicular to the field lines. However, now the orthogonal surfaces (or orthogonal lines in two dimensions) are not closed anymore. Just as electric field lines end where div $\mathbf{E}$ is non-zero (if they end anywhere), the orthogonal surfaces end where curl $\mathbf{E}$ is non-zero (if they end anywhere).

The proof is most easily done in the two-dimensional case. To be concrete let us consider a detail of an electric field whose z component is zero:

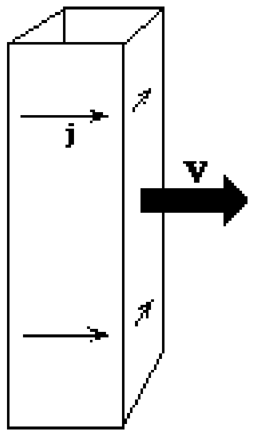

Fig. 4. A "solenoid" consisting of a current carrying sheet moves in a direction perpendicular to its axis. An electric current of current density **j** is flowing in the sheet.

$\mathbf{E}(x, y, z) = (E_x, E_y, 0)$

Then

$\mathbf{S}(x, y, z) = (S_x, S_y, 0) = (E_y, -E_x, 0)$

is an orthogonal field. We calculate the magnitude of curl **E**:

$$|\text{curl}\,\mathbf{E}| = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} = \text{div}\,\mathbf{S}$$

It is seen that the flux sources of the orthogonal field $\mathbf{S}(x, y, z)$ are identical with the circulation sources of the original field $\mathbf{E}(x, y, z)$.

As an example, let us consider the field of Fig. 2c: a superposition of the field of a line charge and a thin solenoid with a changing magnetic flux. In Fig. 3 we not only have represented the field lines but also the orthogonal lines. It is now clearly seen from the figure that at the central spot there is the flux source as well as the circulation source of the field.

Our second example is more intricate and we shall dedicate a separate section to it.

IV. A rectangular electric field

Let us begin by considering a long rectangular current-carrying solenoid. The magnetic induction **B** is parallel to the axis of the solenoid. To simplify the situation, let us replace the windings of the solenoid by a folded current carrying sheet. Moreover, imagine that the sheet is superconducting. So there is no need for an energy source in order to get a persistent electric current. Next, imagine that the "solenoid" is moving with a velocity **v** in a direction perpendicular to its axis, Fig. 4. This is equivalent to considering it in a reference frame which is moving with the velocity $-\mathbf{v}$. Let us call S the reference frame in which the solenoid is at rest and S' the frame in which it is moving. In the new reference frame, in addition to the magnetic field we have a homogeneous electric field. If the magnitude of **v** is small compared to the speed of the light, the field strength of this electric field is

$$\mathbf{E} = \mathbf{v} \times \mathbf{B}$$

Outside of the solenoid, the electric field strength is zero.

Fig. 5 shows the diagram of the electric field with its field lines and orthogonal surfaces. The figure depicts the location of the flux and the circulation sources of the electric field. The flux sources are on those sides of the solenoid, which are parallel to the direction of its movement, the circulation sources are on the faces which are perpendicular to the direction of the movement. Thus, we have a piece of homogeneous electric field with a rectangular cross section and with sharp boundaries.

By admitting the current-carrying sheet to be infinitely thin we skillfully have been sweeping under the carpet the problem of the distribution of the flux and the circulation sources of the electric field in a real conductor. Therefore, let us now consider the current-carrying sheet to have a finite thickness. Since the electric current density is homogeneous over the conductor, the magnitude of the magnetic induction and thus, that of the electric field strength, decreases linearly from its inner to its outer surface. As a consequence, the distributions of the flux sources and that of the circulation sources over the conductor are also homogeneous. The enlargements in Fig. 5 show how the electric field lines and the orthogonal lines are fading away within the conductor when going from the inner to the outer surface.

The explanation of the genesis of the circulation sources of the electric field in the transverse parts of the solenoid is simple: the circulation sources are where the magnetic induction is changing in time. The mechanism of the origination of the flux sources in the lateral parts is due to a relativistic effect [8]: Electric

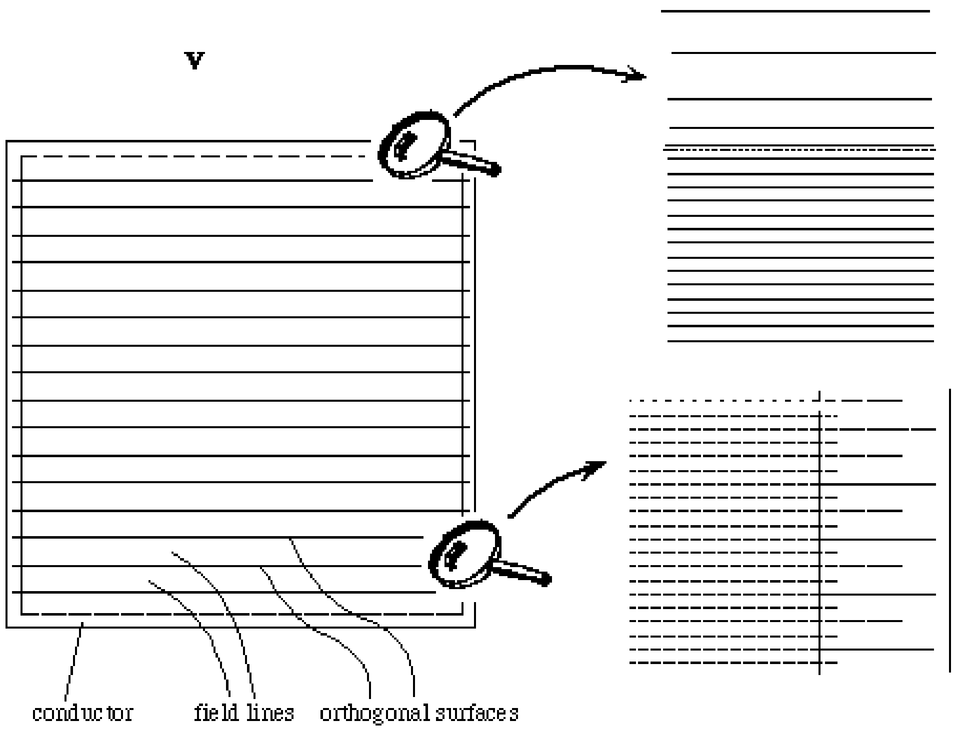

Fig. 5. Field lines and orthogonal surfaces of the rectangular field. It is seen that the flux sources are located on the lateral parts of the sheet (parallel to the direction of the movement) and the circulation sources on the transverse parts (perpendicular to the movement). The enlargements of a section of the transverse part and of a section of the parallel part of the conductor show that the field lines end within the bulk of the parallel parts and the orthogonal lines end within the bulk of the transverse parts.

charge density and electric current density are the components of the current four-vector. In reference frame S the charge density is zero but the current density is non-zero. When changing to reference frame S' the transformation results in a non-zero contribution to the charge density.

VI. Conclusion

In order to represent fields graphically, we propose to draw the orthogonal surfaces in addition to the field lines. The expense is low and the profit high. In such a field diagram one not only distinguishes clearly the distribution of the flux sources but also that of the circulation sources.

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