

Rotational Dynamics and the Flow of Angular Momentum

F. Herrmann and G. Bruno Schmid, Institut für Didaktik der Physik, Universität Karlsruhe, 76128 Karlsruhe, Germany

Abstract

A novel way to introduce angular momentum in an introductory mechanics course is presented. The presentation is by way of a series of experiments with two experiments always appearing together as a pair of analogies: one experiment involving electric charge and the following analogous experiment involving angular momentum. This kind of introduction allows the student, from the very beginning, to become acquainted with the "substancelike" nature of angular momentum, i.e. with the fact that angular momentum can be thought to be distributed in and flow through space. In particular, the proposed approach acquaints the student with the physical nature of angular momentum in mechanics with the same ease with which he becomes acquainted with the properties of electric charge in a traditional approach to electricity theory.

I. Introduction

This paper presents a novel way to introduce angular momentum \mathbf{L} in an introductory mechanics course. Our approach differs from more traditional demonstrations (Connolly and Connolly 1973, Klostergaard 1976, Prigo and Reading 1977, Datta 1978) of angular momentum in its strong orientation toward the electric charge Q . A whole series of experiments are described with two experiments always appearing together as a pair of analogies: one of the experiments involving electric charge and the following, analogous experiment involving angular momentum. This kind of introduction has the advantage that the student becomes acquainted with the "substance-like" nature of angular momentum from the very beginning of the course. Accordingly, the conservation of angular momentum is just as naturally expected by the student as is the conservation of charge. Indeed, no particular effort is necessary to introduce the idea that electric charge is conserved during a typical beginners course in physics whereas the usual introduction of angular momentum conservation via the definition of $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ appears rather contrived.

The far-reaching analogy between electric charge and angular momentum has its basis in the following facts:

- 1) Both are extensive quantities.
- 2) Both are substance-like quantities (Falk et al-in print), i.e. a density and a current density can be associated with each (which is not true for all extensive quantities).
- 3) Both are conserved quantities (which is not true for all substance-like quantities).
- 4) Both are capable of having positive or negative values.

A further characteristic shared by both electric charge and angular momentum which does not, however, play a roll in the following considerations is the fact that each quantity has a universal quantum (e and \hbar , respectively).

The analogy between electric charge and angular momentum is limited by the fact that Q is a scalar whereas \mathbf{L} is a vector. Because of this, we will restrict our considerations of angular momentum throughout this paper to arrangements whereby the angular momentum vector \mathbf{L} and the angular momentum current vector \mathbf{I}_L point along fixed directions in space. This allows us to treat the single, non-zero components of \mathbf{L} and \mathbf{I}_L like scalars. We denote these simply by L and I_L , respectively. Of course, this does not mean the paths by which angular momentum flows in an arrangement are one-dimensional anymore than the paths by which electric charge flows in a network must be one-dimensional.

The ideas presented in this paper are directed toward first semester university students. However, the overall approach is so elementary that it can be adapted without difficulty for presentation to high school students as well. Applications of our ideas to elementary and high school physics teaching have already been published elsewhere (Falk and Herrmann 1981, Schmid 1982).

II. Description of the Course

In this section we present an outline of the course. Many details are left out of discussion where we feel these are obvious and simple enough to be filled in by the reader himself, for example, figures to simple experiments and all questions concerning units. In addition, only those features which we consider novel (and not the entire discussion of angular momentum actually presented in the course such as considerations of the moment of inertia, the parallel-axis theorem, etc.) are presented here.

Most of the material necessary for the experiments presented in this paper should be available in the stock of any well-equipped college. For the mechanical experiments, two flywheels are needed which can be either held by hand or fastened to a table top. The flywheels can be spun up either directly by hand or indirectly with, say, the edge of a disc sander fitted to an electric drill. For some demonstrations, the flywheels must be connected via a friction coupling. To this end, a short piece of garden hose can be affixed to one flywheel concentric to its axis and a bottle cork (which can be inserted into the hose) to the other flywheel (Figure 1a). For one experiment (Section II.11), the flywheels must be connected via a torsion spring. A clock spring serves particularly well for this purpose. Such a spring can be outfitted so that it automatically decouples when it is not under tension (Figure 1b). The arrangement for the experiment in Section II.3 can be easily built with the help of a construction kit.

For the electrical experiments, the most frequently used elements are capacitors. A circuit is needed which lights a lamp upon the discharging of a capacitor. Thus, the capacitance should be of the order of several mF. Accordingly, it will be sufficient to charge the capacitor up to about 10 volts. The inductance of the coil used in the experiment of Section II.11 should be on the order of a few H so that the oscillations are slow enough to be followed easily on a voltmeter. In order that the capacitors are not significantly discharged upon connecting a voltmeter across them, the voltmeter must have a sufficiently high internal resistance, for example, a voltmeter with a preamp could be used.

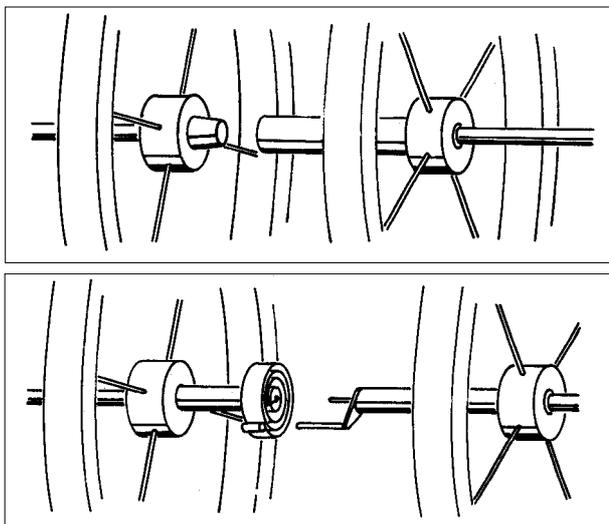


Fig. 1 a) Two flywheels which can be axially connected across a friction coupling by pushing them together.

b) Two flywheels which can be axially connected across a torsion spring. The flywheels decouple as as soon as the spring is relaxed.

II.1 Angular Momentum as Spin

A sphere which has been charged with the help of, say, a van-de-Graaf generator “has” electricity. If the sphere is well insulated, it keeps this electricity. If we carry the sphere around the room, we are also carrying electricity around the room. A physicist calls electricity 'electric charge'.

A flywheel which has been spun up with the help of, say, a motor “has” spin. If the axle of the flywheel is fitted with good bearings, it keeps this spin. If we carry the flywheel around the room, we are also carrying spin around the room. A physicist calls spin 'angular momentum'.

II.2 Angular Momentum Currents

If we touch an electrically charged sphere to the ground, the sphere will discharge: Electric charge flows into the earth.

If we touch a spinning flywheel to the ground, the flywheel will stop spinning: Angular momentum flows into the earth.

If we connect an electrically charged sphere with a wire across a resistor to an uncharged sphere, the amount of electric charge possessed by the one sphere decreases while that of the other increases. We conclude from this that an electric charge current flows from the one sphere through the wire and resistor into the other sphere. To aid in demonstrating this flow of electric charge, it is helpful to use an object which can hold much more electric charge than a sphere, namely, a commercial capacitor. We can repeat the above experiment with each sphere now replaced by a capacitor. To make evident the flow of electric charge through the wire joining the capacitors, the wire can be connected across an electric light bulb (Figure 2a). The flow of electric charge through the wire joining the capacitors is evidenced by the fact that the bulb lights up.

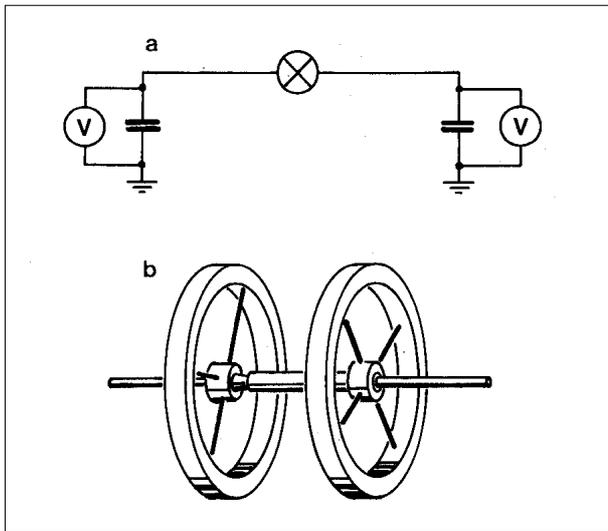


Fig. 2 a) Two capacitors connected across a light bulb. Initially, the one capacitor is charged and the other uncharged. Upon the flow of electric charge from the one capacitor to the other, the light bulb lights up.

b) Two flywheels connected across a friction coupling. Initially, the one flywheel is spinning and the other is at rest. Upon the flow of angular momentum from the one flywheel to the other, the friction coupling heats up.

If we connect a spinning flywheel with a shaft across a friction coupling to another flywheel at rest, the amount of angular momentum possessed by the one flywheel decreases while that of the other increases. We conclude from this that an angular momentum current flows from the one flywheel through the shaft and friction coupling into the other flywheel (Figure 2b). The flow of angular momentum through the shaft joining the flywheels is evidenced by the fact that the friction coupling heats up.

Consider two identical capacitors. Let them be equally but oppositely charged and connect them with a wire across a resistor. It is easy to determine with the help of a voltmeter that, after waiting a short while, both capacitors are completely discharged. We see that electric charge can take positive as well as negative values such that equal amounts positive and negative charge add to zero.

Consider next two identical flywheels. Let them be equally but oppositely spun and connect them with a shaft across a friction coupling. It can be directly seen that, after waiting a short while, both flywheels are completely at rest. We see that angular momentum can take upon positive as well as negative values such that equal amounts of positive and negative angular momentum add to zero.

II.3 Angular Momentum Conductance

Consider charging a capacitor with a voltage stabilized power supply in each of several different networks. Let a different element be built into each respective network in addition to the capacitor and charging device and follow the charging process on a voltmeter. The time required to charge up the capacitor to some arbitrary potential in each case depends upon the nature of the extra element built into the respective network. For example, if this extra element is a piece of copper wire the charging process goes quickly; if the extra element is a salt bath (Figure 3a), more time is required; if this extra element is simply an air gap, i.e. if the circuit is open, the capacitor does not charge up at all. From this we conclude that copper is a good conductor of electric charge, salt water is not such a good conductor and air is a nonconductor of electric charge. Whether an object is a good or a poor conductor of electric charge depends (among other things) upon its material composition. The ability of a material to conduct electric charge is described in terms of the material's conductivity.

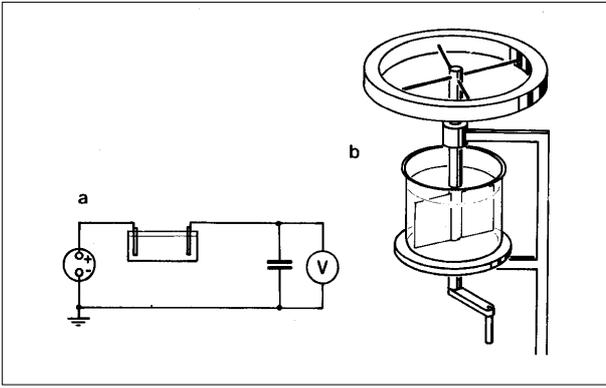


Fig. 3 a) An electrical network used to charge up a capacitor. The bath of salt water built into the circuit as a conductor delays the charging process.

b) A mechanical network used to spin up a flywheel. The bath of oil built into the circuit as a friction coupling delays the spinning process.

Consider spinning a flywheel from rest with a crank, i.e. charging a flywheel with angular momentum by means of a crank. The crank is rotated at a fixed angular rate throughout the process. Let any one of several different elements be built into the shaft of the crank and observe the rate by which the flywheel can be spun up from rest in each case. The time required to spin the flywheel up (to some arbitrary angular velocity) in each case depends upon the nature of the extra element built into the shaft. For example, if this extra element is a solid object, the spinning process goes quickly; if the extra element is a friction coupling, say, an oil bath (Figure 3b), more time is required; if the shaft is open, the flywheel does not spin up at all. From this we conclude that a solid rod is a good conductor of angular momentum, thick oil is not such a good conductor and air is a nonconductor of angular momentum. Whether an object is a good or a poor conductor of angular momentum depends (among other things) upon its material composition. The ability of a material to conduct angular momentum is described in terms of the material's viscosity.

II.4 Measurement of the Angular Momentum Current

If we send an electric current through a coil, a magnetic field is created in the coil. This magnetic field can be used to measure the strength of the electric current flowing through the wire of the coil (Figure 4a).

If we send an angular momentum current through an elastic shaft (or bar), the shaft will be twisted. This twist, namely, the angle through which the shaft is twisted, is a measure of the strength of the angular momentum current flowing through it (Figure 4b). The (negative) angular momentum current is also called 'torque'.

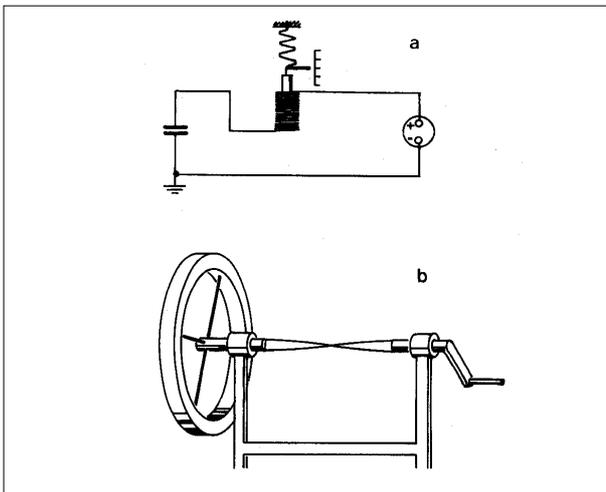


Fig. 4 a) Measurement of the electric current strength via the magnetic field produced in a coil.

b) Measurement of the angular momentum current strength via the twist produced in a ribbon of steel.

II.5 The Continuity Equation for Angular Momentum

If the amount of electric charge Q contained in a region of space R is changing, a net electric charge current I_Q must be flowing through the boundary surface of R (Figure 5a)

$$\frac{dQ}{dt} + I_Q = 0 \quad (1)$$

Equation (1) is the continuity equation in integral form for electric charge.

If the amount of angular momentum L contained in a region of space R is changing, a net angular momentum current I_L must be flowing through the boundary surface of R (Figure 5b)

$$\frac{dL}{dt} + I_L = 0 \quad (2)$$

Equation (2) is the (one-dimensional) continuity equation in integral form for angular momentum.

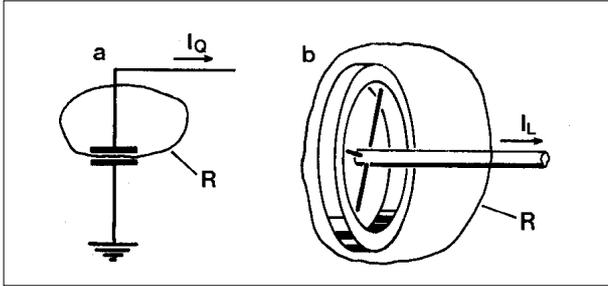


Fig. 5 a) Illustration of the continuity equation for electric charge (Equation (1) of text).
b) Illustration of the continuity equation for angular momentum (Equation (2) of text).

II.6 Angular Momentum Equilibrium

Consider two different capacitors with different potential differences across each. Let them be connected together via a resistor (like Figure 2a) and measure the potential drop across each capacitor as a function of time. An electric current continues to flow through the resistor until the voltages across both capacitors are equal. After the electric current ceases to flow, we say the two capacitors are in electrical equilibrium.

Now consider two different flywheels spinning at different angular velocities. Let them be connected together via a friction coupling (Figure 2b) and observe the angular velocities of the two flywheels as a function of time. An angular momentum current continues to flow through the friction coupling until the angular velocities of both flywheels are equal. After the angular momentum current ceases to flow, we say the two flywheels are in rotational equilibrium.

II.7 Angular Momentum Capacity

Consider two identical capacitors. Let one of these be charged up until the potential difference across it is U and the corresponding amount of charge on each plate is Q . Then bring this capacitor into equilibrium with the other, initially uncharged capacitor. Because of the symmetry of the network, the charge of each capacitor at equilibrium will be $Q/2$. The corresponding potential drop across each capacitor can be measured and will be found to be $U/2$. The charge Q could just as well be distributed over an arbitrary number of identical capacitors and the potential drop across each could be measured. Regardless of the number of capacitors used, the measured potential drop across each capacitor is proportional to the charge of the capacitor: $U \propto Q$. Differently constructed capacitors, however, will not have the same charge at equilibrium, i.e. at the same potential drop. The proportionality factor C between Q and U is called the 'capacitance' of the capacitor. The capacitance is a measure of the charge of a capacitor for a given potential drop across it.

Consider now two identical flywheels. Let one of these be spun up until its angular velocity is ω and its corresponding angular momentum is L . Then bring this flywheel into equilibrium with the other, initially resting flywheel. Because of the symmetry of the arrangement, the angular momentum of each flywheel at equilibrium will be $L/2$. The corresponding angular velocity of each flywheel can be measured and will be found to be $\omega/2$. The angular momentum L could just as well be distributed over an arbitrary number of identical flywheels and the angular velocity of each could be measured. Regardless of the number of flywheels used, the measured angular velocity of each flywheel is proportional to the angular momentum of the flywheel: $L \propto \omega$. Differently constructed flywheels, however, will not have the same angular momentum at equilibrium, i.e. at the same angular velocity. The proportionality factor J between L and ω is cal-

led the 'moment of inertia' of the flywheel. The moment of inertia is a measure of the angular momentum of a flywheel for a given angular rate of rotation.

II.8 Energy transfer through rotating shafts

Energy can be transferred from a battery to a motor (Figure 6a) with the help of a wire. In the process, an electric charge current flows in a closed loop: from the battery through the wire to the motor and back again to the battery through the earth. The energy current (or "power") I_E depends both upon the potential difference U between the wire and the earth as well as upon the electric charge current I_Q according to the relation

$$I_E = U \cdot I_Q.$$

If the motor is shorted out of the circuit, $U = 0$. Accordingly, $I_E = 0$ even though an electric charge current I_Q continues to flow through the network. If, on the other hand, the circuit is opened at the motor so that $I_Q = 0$, I_E will vanish even though $U \neq 0$.

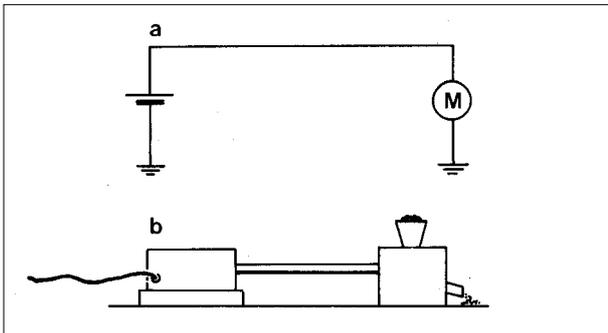


Fig. 6 Illustration of the the transfer of energy in (a) an electrical or (b) a rotational system.

a) Energy is being transferred from a battery to a motor with the help of a wire.

b) Energy is being transferred from a motor to a coffee grinder with the help of a shaft.

Energy can be transferred from a motor to a coffee grinder (Figure 6b) with the help of a shaft. In the process, an angular momentum current flows in a closed loop: from the motor through the shaft to the coffee grinder and back again to the motor through the earth. The energy current I_E depends both upon the difference ω in angular velocity between the shaft and the earth as well as upon the angular momentum current I_L according to the relation

$$I_E = \omega \cdot I_L.$$

If the coffee grinder is "shorted out", i.e. if the end of the shaft is rigidly connected to the earth, then $\omega = 0$. Accordingly, $I_E = 0$ even though an angular momentum current I_L continues to flow through the motor, shaft and earth. (Traditionally, one speaks of a torque acting in the motor, shaft and earth in this case). If, on the other hand, the shaft is allowed to rotate freely so that $I_L = 0$, i.e. so that the torque in the shaft vanishes, I_E will vanish even though $\omega \neq 0$.

In Figure 6a, one commonly says that energy is being transferred in the form of electric energy from the battery to the motor. However, we prefer to say in this case that energy is being "carried" (Falk et al-in print, Schmid 1982) by electric charge from the battery to the motor since the term "electric energy" suggests there is more than one physical quantity: energy.

In Figure 6b, one commonly says the motor is doing work on the coffee grinder. However, we prefer to say in this case that energy is being "carried" by angular momentum from the motor to the coffee grinder since the term "work" covers up the fact that the occurrence being dealt with is an energy transfer process, i.e. that the relevant physical concept is that of an energy flow. Furthermore, speaking about work instead of an energy current does not bring out the analogy between this example and the one illustrated in Figure 6a (as well as several other analogies which are traditionally hidden under different names).

In Figure 6a, the electric potential is a measure of how much energy the energy carrier current I_Q is "loaded with". In the same way, in Figure 6b, the angular velocity is a measure of how much energy the energy carrier current I_L is loaded with.

II.9 The Gear Box: An Angular Momentum Transformer

An energy current flows through an electric transformer simultaneously with the flow of an energy carrier current, the electric charge current I_Q . At the one side of the transformer, the carrier current I_Q is large and the energy load factor U is small. At the other side of the transformer, the carrier current is small and the energy load factor is large (Figure 7a).

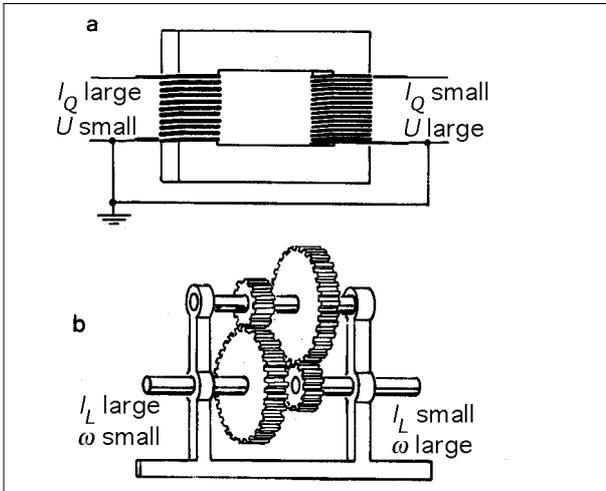


Fig. 7 a) Electric transformer.

b) Gear Box, an angular momentum transformer.

An energy current flows through a gear box simultaneously with the flow of an energy carrier current, the angular momentum current I_L . At the one side of the gear box, the carrier current I_L is large and the energy load factor ω is small. At the other side of the gear box, the carrier current is small and the energy load factor is large (Figure 7b).

II.10 Flywheels and Torsion Springs as Energy Containers

While an electric charge current is flowing onto a capacitor plate, an energy current is flowing into the capacitor. Energy is being stored in the capacitor. The amount E of stored energy can be calculated by integrating the net inflowing energy current I_E over time:

$$E = \int_0^t I_E dt = \int_0^t UI_Q dt = C \int_0^U U dU = \frac{C}{2} U^2$$

Here use has been made of the relation $I_Q = C \cdot U / t$

(from $Q = C \cdot U$) between the charge current and the time-rate-of-change U / t of the voltage across the capacitor. C is the capacitance of the capacitor.

While a voltage is being applied across a coil, an energy current is flowing into the coil. Energy is being stored in the coil. The amount E of stored energy can be calculated according to

$$E = \int_0^t I_E dt = \int_0^t UI_Q dt = L \int_0^{I_Q} I_Q dI_Q = \frac{L}{2} I_Q^2$$

Here use has been made of the relation $U = L \cdot I_Q / t$ between the potential difference U across the coil and the time-rate-of-change I_Q / t of the electric charge current I_Q flowing through the coil. L is the inductance of the coil.

While an angular momentum current is flowing into a flywheel, an energy current is flowing into the flywheel. Energy is being stored in the flywheel. The amount E of stored energy can be calculated by integrating the net inflowing energy current over time:

$$E = \int_0^t I_E dt = \int_0^t \omega I_L dt = J \int_0^\omega \omega d\omega = \frac{J}{2} \omega^2$$

Here use has been made of the relation $\omega = J \cdot \omega' / t$ (from $L = J \cdot \omega$) between the angular momentum current I_L and the time-rate-of-change ω' / t of the angular velocity. J is the moment of inertia of the flywheel.

While an angular velocity difference “is applied” to the ends of a torsion spring, an energy current is flowing into the spring. Energy is being stored in the torsion spring. The amount E of stored energy can be calculated according to

$$E = \int_0^t I_E dt = \int_0^t \omega I_L dt = D \int_0^{I_L} I_L dI_L = \frac{D}{2} I_L^2$$

Here use has been made of the relation $\omega = D \cdot I_L / t$ (from $\alpha = D \cdot I_L$, where α is the angular displacement) between the angular velocity difference ω across the spring and the time-rate-of-change I_L / t of the angular momentum current I_L flowing through the spring. D is the inverse rotational compliance of the torsion spring.

II.11 Oscillations: The Torsion Pendulum

Consider two capacitors with the same capacitance connected across a switch through a coil. Let the charge of each capacitor be monitored with two separate voltmeters (Figure 8a). Assume an initial state with the switch open: one capacitor, say, capacitor 1 is charged up and the other capacitor (labelled 2) is uncharged. After the switch is closed, oscillations can be observed in the readings of both voltmeters. These readings indicate harmonic oscillations in the electric charge of the plates of each capacitor, in the strength of the electric current flowing between the capacitors, in the electric potential difference across each capacitor, in the energy stored within each capacitor and in the flow of energy between the capacitors.

Consider now two flywheels with the same moment of inertia. The flywheels can be connected along their axes through a torsion spring (Figure 8b). Assume an initial state with the flywheels disconnected: one flywheel, say, flywheel 1 is spinning and the other flywheel (labelled 2) is at rest. After the flywheels are connected, harmonic oscillations can be observed in the angular momentum of each flywheel, in the strength of the angular momentum current (torque) flowing between the flywheels, in the angular velocity difference between each flywheel and the earth, in the energy stored within each flywheel and in the flow of energy between the flywheels.

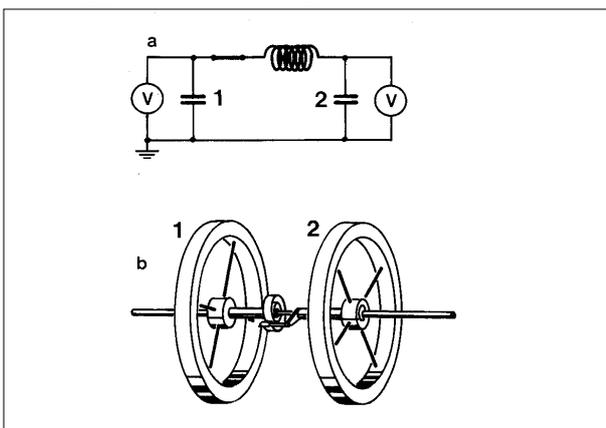


Fig. 8 a) Two capacitors with the same capacitance connected across a switch through a coil. The charge of each capacitor can be monitored with two separate voltmeters.

b) Two flywheels with the same moment of inertia connected along their axes through a spring.

If a rectifier is built into the circuit of Figure 8a, the oscillations in the network last for only half a period. At the end of this time, capacitor 2 is charged, capacitor 1 is discharged.

If the flywheels of Figure 8b are coupled along their axes so that they automatically decouple whenever the angular momentum current changes direction, the oscillations in the arrangement last for only half a period. At the end of this time, flywheel 2 is spinning, flywheel 1 is at rest. (This process is sometimes called an “elastic collision”. The approach to equilibrium mentioned above in Section II.6 is called an “inelastic collision”).

II.12 “Charged” and “Uncharged” Conductors of Angular Momentum

When an electric current is flowing through a wire, the wire is almost electrically neutral, i.e. the positive and negative charge within the wire compensate one another. However, there are also electric currents with a large non-zero charge density, for example, the flow of electric charge with a current of electrons in a television tube.

When an angular momentum current is flowing through a rod, the angular momentum of the rod is almost zero. However, there are also angular momentum currents with a large non-zero angular momentum, for example, the flow of angular momentum with a current of polarized neutrons in an accelerator arrangement.

A current of polarized electrons represents both an electric charge current and an angular momentum current (plus currents of several other substance-like quantities as well).

III. The Choice of Analogy

An analogy of the type presented above in Section II can be just as well carried out between electric charge and linear momentum p . This might already have been expected, since the analogy between angular momentum and linear momentum is well-known. Indeed, we have introduced linear momentum in exactly this way in a beginners mechanics course at the University of Karlsruhe. Of course, one difficulty within this presentation is inherent to the fact that, unlike electric charge or angular momentum, a body “charged” with linear momentum cannot be simply held in one's hands and carried around the lecture hall: The nature of a body containing linear momentum is to move across the room on its own!

In a certain sense, it would be more physically satisfying to choose the following analogy: electric charge \leftrightarrow linear momentum and electric dipole moment \leftrightarrow angular momentum. A comparison of the electric dipole moment with angular momentum is, on the one hand, very insightful especially concerning discussions of the angular momentum density. Indeed, the dipole moment density is an established quantity well-known under the name of “electric polarization”. On the other hand, this comparison is not so far-reaching since the electric dipole moment is not a conserved quantity like angular momentum.

Finally, we would like to make a few comments about other more familiar analogies between mechanics and the theory of electricity as these are usually referred to or systematically presented in many text books (Olson 1958, MacFarlane 1964). These do not, as we do, compare Q with L (or Q with p) and U with ω (or U with the linear velocity v). However, because of the logical symmetries within mechanics and electrodynamics themselves, these analogies are, from a formal standpoint, no less justifiable than the one presented in this paper. Nevertheless, the usual analogies suffer from a major conceptual drawback: The conserved substance-like quantity Q does not correspond to the conserved substance-like quantity L but, rather, to the angle α to which there is neither a current ($d\alpha/dt$ is no angle current) nor a density, nor a conservation law.

IV. Conclusions

The analogy between electric charge and angular momentum presented in this paper is advantageous:

- 1) For the economy of thought it introduces into one's understanding of how angular momentum and electric charge can be treated in mechanics and electrodynamics. Economy of thought has always been an important aspect to every physical analogy.
- 2) For the conceptual clarity it introduces into one's understanding into the physical nature of angular momentum.

The traditional introduction of angular momentum \mathbf{L} via the equation $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ does not bring out the two main points emphasized by the analogy presented in this paper, namely, (i) that angular momentum is a physical quantity of its own, which in general is independent of the linear momentum \mathbf{p} and (ii) that angular momentum is just as substance-like in nature as electric charge (or as mass or as amount of substance or as entropy). Emphasizing these two points upon the introduction of angular momentum as a physical quantity enables one to have just as firm an intuitive grasp upon angular momentum as one traditionally has upon electric charge.

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