The Poynting vector field and the energy flow within a transformer

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(Received 6 June 1984; accepted for publication 31 May 1985)

The Poynting vector field of a transformer with two separated coils and long, parallel arms has the same distribution as that of a pair of parallel electric conductors. The magnetic field between the arms of a transformer plays the same important role for the energy transport in a transformer as does the electric field for the transport of energy between the conductors of a conductor pair. In the latter case, the energy current (= power) P is given by the familiar expression $P = UI$, where U is the electric tension between the conductors and I the electric charge current through them. In the former case, we find an analogous expression $P = U_m I_m$ for the energy current in a transformer. Here U_m is the magnetic tension $U_m = \int d\mathbf{r}\, \mathbf{H}$ between the arms of the transformer ($\mathbf{H}$ = magnetic field vector) and $I_m = -\int d\mathbf{A}\, \dot{\mathbf{B}}$ is the Hertz magnetic current ($\dot{\mathbf{B}}$ = the time derivative of the magnetic induction $\mathbf{B}$) through them. An experiment will be described which shows that the energy loss in a transformer is related to a magnetic potential drop within each of the two arms of the transformer.

I. INTRODUCTION

When energy is transmitted via an electromagnetic field, the energy current P can be calculated according to

$$P = \int d\mathbf{A}\, \mathbf{S},$$

where $\mathbf{S}$ is the cross product of the electric and magnetic field vectors

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}. \tag{1}$$

It is helpful to distinguish between several distinct cases of energy transport in the electromagnetic field depending upon the boundary conditions:

(1) The electric and magnetic fields are not coupled to sources, i.e., $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$. In this case, one speaks of electromagnetic waves. This is the most frequently discussed example for applications of the Poynting vector.[1–4]

(2) The electric field has sources, i.e., $\nabla \cdot \mathbf{E} \neq 0$ and the magnetic field does not, i.e., $\nabla \cdot \mathbf{H} = 0$. This case is realized, e.g., when the fields are guided by conductors, e.g., by a waveguide. The Poynting vector field is also discussed in some textbooks[4] for this example. An extreme example of a waveguide is the familiar two-wire electric cable. Such a "waveguide" is useful if the frequency of the fields is zero or small. Since, later in this paper, we will discuss the Poynting vector field of a two-wire resistance-free cable, qualitative sketches of the $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{S}$ fields for such a case are given in Fig. 1. Energy flows in the immediate vicinity outside the conductors and parallel to them.

(3) The magnetic field has sources, i.e., $\nabla \cdot \mathbf{H} \neq 0$, and the electric field does not, i.e., $\nabla \cdot \mathbf{E} = 0$. In this case, the fields are guided by magnetized material.

(4) The electric and magnetic fields are both coupled to sources, i.e., $\nabla \cdot \mathbf{E} \neq 0$ and $\nabla \cdot \mathbf{H} \neq 0$. (And there are no electric or magnetic fields present without sources.) In this case, the energy current density results from the crossed electric and magnetic fields of charged bodies and, say, permanent magnets, respectively. Because one now has closed S-field lines, this case is of the least interest for practical applications, but of some pedagogic interest regarding the validity of expression (1).

In this paper, we are concerned exclusively with case (3) above. This case shows up when discussing electric devices comprising a soft iron core, e.g., generators, electric motors, and transformers. In particular, we will discuss the transport of energy in an arrangement with an especially simple geometry: a long, stretched-out transformer (see Fig. 2). The separation between the primary and secondary coils is taken to be much larger than the separation between the "upper" and "lower" arms of the iron core.

It will be shown that there is a far-reaching analogy between the long upper and lower arms of such a core and the two wires of a typical electric cable (see Fig. 1).

In Sec. II the configuration of the electric, magnetic, and Poynting vector fields of a transformer are given qualitatively. In Sec. III, the integral quantities, magnetic tension U_m, and magnetic current I_m are introduced and the power of the transformer is expressed as a function of these quantities. Finally, in Sec. IV, two experiments will be described. The first experiment allows one to measure the physical quantities characteristic of the energy current in a transformer as a function of the load on the transformer. The second experiment shows that the energy loss in a transformer is related to the magnetic potential drop along each of the two arms of the transformer.

In preparing the final draft of this manuscript, the paper by Newcomb[5] came to our attention. This paper is a reply to an earlier Letter to the Editor by Siegmann[6] asking

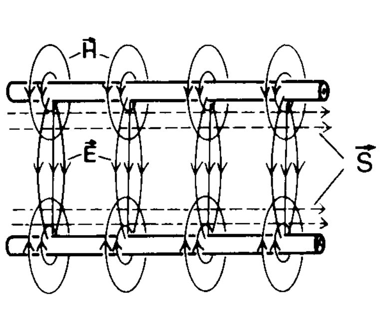

Fig. 1. Qualitative sketch of the E, H, and S fields for a two-wire resistance-free cable.

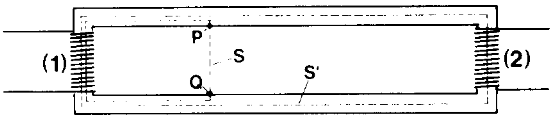

Fig. 2. Sketch of a transformer with elongated arms between the two separated coils. The separation between the primary and secondary coils is much larger than the separation between the upper and lower arms of the core. S and S' label two different paths of integration.

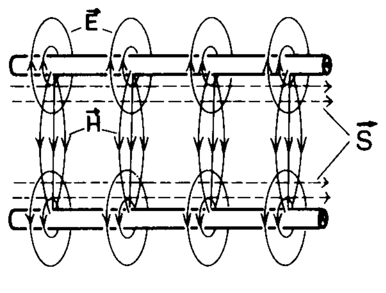

Fig. 3. Qualitative sketch of the **E**, **H**, and **S** fields between the arms of the transformer.

about the Poynting vector field of an ideal transformer. Our paper differs from Newcomb's response insofar as we introduce the analogy between the energy transport accompanied by the flow of electric or magnetic currents.

II. THE ELECTRIC, MAGNETIC, AND POYNTING VECTOR FIELDS OF AN ELONGATED TRANSFORMER

Consider a transformer of the shape illustrated in Fig. 2. We label those quantities which refer to the primary circuit with a number (1) and those which refer to the secondary circuit with the number (2).

For an ideal transformer under a load,[7] the electric current is in phase with the electric tension both in the primary as well as in the secondary winding. (For this reason, all symbols given below refer, where appropriate, to the instantaneous values of the corresponding quantities.) We then have

$$U(1)/N(1) = U(2)/N(2) \tag{2}$$

and

$$N(1)\cdot I(1) \cong N(2)\cdot I(2) \tag{3}$$

Here $U(i)$, $N(i)$, and $I(i)$ are the electric tension, the number of turns, and the electric current, respectively, in the winding i.

The electric current which flows at no load (sometimes called the "exciting current") and which produces the magnetic flux within the iron core linking both windings (sometimes called the "mutual flux") is small compared to a typical current in the primary under load. The mutual flux is nearly the same with or without a load although, under load, the current in the primary is increased. The effect of the additional load current in the primary on the mutual flux is nearly exactly compensated by the load current in the secondary, i.e., the load currents in the primary and secondary windings create magnetic fluxes of opposite direction in the core such that the mutual flux remains essentially constant.

Applying Ampere's law to the paths S and S' sketched in Fig. 2:

$$\oint_S d\mathbf{r}\,\mathbf{H} = N(1)\cdot I(1), \tag{4}$$

$$\oint_{S'} d\mathbf{r}\,\mathbf{H} = N(1)\cdot I(1) - N(2)\cdot I(2). \tag{5}$$

In view of Eq. (3), the integral in Eq. (5) is much smaller than the integral in Eq. (4). Furthermore, since the magnitude of the **H** field within the core of the transformer is very much smaller than in the region between the arms of the transformer, the integral in Eq. (4) is practically equal to the integral of **H** along the path between the points P and Q in Fig. 2, so that

$$N(1)\cdot I(1) \cong \int_P^Q d\mathbf{r}\,\mathbf{H}. \tag{6}$$

The **H** field in the region between the transformer arms has sources ($\nabla\cdot\mathbf{H} = -\nabla\cdot\mathbf{M}$, $\mathbf{M}$ = magnetization of the core) on one arm and sinks on the other. (See Fig. 3.) The direction of the **H** field changes in phase with the current $I(1)$.

Let us now apply Faraday's law to an arbitrary area A penetrated by one arm of the transformer:

$$\oint d\mathbf{r}\,\mathbf{E} = -\int_A d\mathbf{A}\,\dot{\mathbf{B}} = U(1)/N(1). \tag{7}$$

Here $\dot{\mathbf{B}}$ is the time rate of change of the magnetic induction **B**. According to Eq. (7), each arm of the transformer is surrounded by closed **E** field lines. (See Fig. 3.)

Since the current and tension in the coils of the transformer are in phase, the **E** and **H** fields are also in phase.

Notice that the **E** and **H** fields in the transformer have the same configuration as the **H** and **E** fields, respectively, of a pair of current-carrying conductors with the same geometry (up to the sign of either field). Thus the question about the Poynting vector field can be immediately answered: *The Poynting field of the transformer has the same configuration as that of a pair of current-carrying conductors with the same geometry.*

It is noteworthy that the **H** field outside the iron core of a transformer (which one might be inclined to regard as an undesirable side effect) is responsible for energy transport within the transformer—just as necessary for energy transport as is the electric field between the conductors of a charge-carrying conductor pair.

III. MAGNETIC TENSION AND MAGNETIC CURRENT STRENGTH

It is convenient to express the energy current (or "power") provided by an electric network in terms of the integral quantities U (electric tension) and I (electric current) as

$$P = U\cdot I. \tag{8}$$

For the same reason, we would also like to express the energy current flowing through a transformer in terms of integral quantities. Since we assume our transformer operates without losses, the energy current flowing through the transformer must be equal to the energy currents flowing into as well as out of the device:

$$P = U(1)\cdot I(1) = U(2)\cdot I(2). \tag{9}$$

[The second equality in Eq. (9) also follows directly from Eqs. (2) and (3).]

With the help of Eqs. (6) and (7), $I(1)$ and $U(1)$ can be replaced in Eq. (9), resulting in

$$P = \left(\int_P^Q d\mathbf{r}\, \mathbf{H}\right) \cdot \left(-\int d\mathbf{A}\, \dot{\mathbf{B}}\right). \quad (10)$$

Equation (10) allows for a simple interpretation: The first factor represents the magnetic tension U_m between the two arms of the transformer

$$U_m = \int_P^Q d\mathbf{r}\, \mathbf{H} \quad (11)$$

and the second integral is identified as the magnetic current strength I_m:

$$I_m = -\int d\mathbf{A}\, \dot{\mathbf{B}}. \quad (12)$$

This concept is introduced in analogy to Maxwell's displacement current $\int d\mathbf{A}\, \dot{\mathbf{D}}$, where $\dot{\mathbf{D}}$ is the time rate of change of the electric displacement $\mathbf{D}$. It has already been introduced by Hertz[8] and was later used by Born.[9] However, as opposed to the concept of magnetic tension, the idea of magnetic current strength did not establish itself. One reason for this might be that I_m is not associated with the motion of free-magnetic charge. Of course, the construction of a current is not logically dependent upon whether or not there exists a microscopic mechanism whereby some kind of particles move with a well-defined velocity.

Inserting Eqs. (11) and (12) in Eq. (10), we obtain the following expression for the energy current flowing in a transformer:

$$P = U_m I_m. \quad (13)$$

The analogy to Eq. (8) for the energy current flowing in a two-wire electric cable is obvious. Accordingly, it is suggestive to call the two arms of a transformer "magnetic conductors." A good magnetic conductor is a "magnetically soft" material with a high permeability, e.g., iron. In terms of this analogy, Eq. (13) can be expressed in the following way: The energy current flowing between a pair of magnetic conductors is equal to the product of the magnetic tension between the conductors times the strength of the magnetic current flowing through them.

Recapitulating Eqs. (3), (6), and (11), as well as Eqs. (2), (7), and (12),

$$N(1) \cdot I(1) = U_m = N(2) \cdot I(2), \quad (14)$$

$$U(1)/N(1) = I_m = U(2)/N(2). \quad (15)$$

From these equations, one can recognize two interesting characteristics of a transformer: The magnetic tension depends only upon the strength of the electric current and the strength of the magnetic current depends only upon the electric tension. If the transformer is connected to a voltage stabilized alternating current source, e.g., to a commercial network, then the magnetic current in the iron core is independent of the load. A change in load results simply in a change in the magnetic tension between the upper and lower magnetic conductors. On the other hand, if the amplitude of the electric current is stabilized in the primary winding, then the magnetic tension is independent of the load. A change in the resistance provided by the load causes a change only in the magnetic current strength.

IV. DEMONSTRATION EXPERIMENTS

The apparatus used in the following experiments can be found in stock at most schools or universities. For the sake of clarity, we give specific data about the number of windings, inductivity, and ohmic resistance although, of course, these need not be followed exactly to carry out a successful experiment.

A. Demonstration

The first thing to demonstrate is the validity of Eqs. (14) and (15): (1) The relation between the magnetic tension U_m between the upper and lower arms of the transformer and the electric currents $I(1)$ and $I(2)$ in the primary and secondary windings, respectively; and (2) the relation between the magnetic current I_m in the iron core of the transformer and the electric tensions $U(1)$ and $U(2)$ in the primary and secondary windings, respectively.

A transformer is constructed with the following materials: (1) One U-shaped piece of laminated iron (cross section 4×4 cm), (2) one straight piece of laminated iron, and (3) two coils with $N(1) = N(2) = 500$ and $R(1) = R(2)$ (see Fig. 4). A variable resistor is connected to the secondary winding as the load. A voltmeter and amperemeter are connected into both the primary and secondary circuits.

In addition to the electric quantities, we also want to measure the magnetic quantities U_m and I_m. Of course, we are only interested in the changes of these quantities under changes in the load and not, however, in their absolute values. Therefore, it suffices to determine the values of U_m and I_m up to a constant factor.

For the measurement of I_m, a small coil with $N = 10$ windings is wrapped around one of the magnetic conductors and connected to a voltmeter. The reading on the voltmeter is proportional to I_m according to the relation $U = -N \int d\mathbf{A}\, \dot{\mathbf{B}} = N \cdot I_m$.

U_m is measured with a commercial Hall probe. The device must be able to register alternating fields. The most sensitive setting should correspond to a full range of 10 mT (such devices are typically normed in T and not in A/m). If the Hall probe is installed somewhere in the region between the magnetic conductors, the reading is proportional to the magnetic field strength $\mathbf{H}$ at the location of the probe and, therefore, is proportional to $U_m = \int d\mathbf{r}\, \mathbf{H}$.

An alternating tension of $U = 50$ V is now applied to the primary winding of the transformer. If the transformer were to have an efficiency of 100%, the instruments would indicate $U(1) = U(2)$ and $I(1) = I(2)$. Measurements show that $U(2)$ is almost as large as $U(1)$, whereas $I(2)$ is

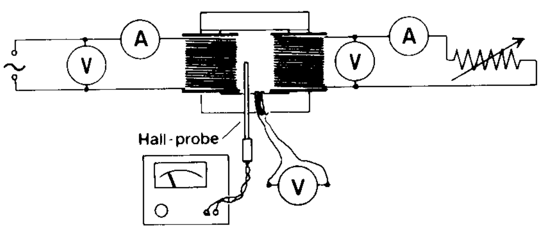

Fig. 4. Sketch of the experimental setup for demonstrating the validity of Eqs. (14) and (15).

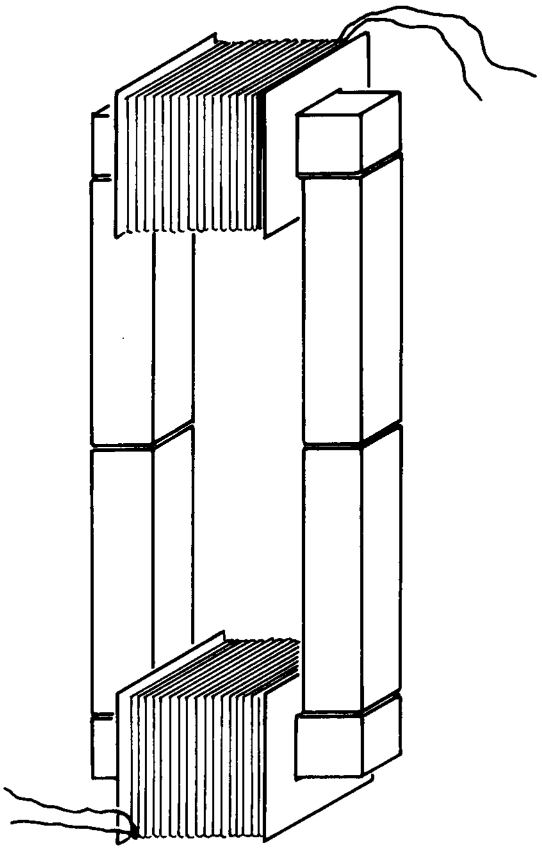

Fig. 5. Sketch of the experimental setup for demonstrating the existence of a magnetic potential drop along the magnetic conductors of a transformer.

clearly smaller than $I(1)$. One must, in fact, be satisfied with a maximum efficiency of around 80%.

We now vary R between 100 and 300 Ω (we should not come too close to a short) and notice, as expected, that $U(1)$ and $U(2)$ remain constant while $I(1)$ and $I(2)$ change in proportion to one another. The behavior of the U_m and I_m readings is more interesting: The magnetic current I_m, like $U(1)$ and $U(2)$, remains virtually constant under changes of R while the magnetic tension U_m changes to the same extent as the electric current. This confirms relations (14) and (15).

B. Demonstration

We construct a long transformer with the same coils as above and six straight laminated iron pieces. (See Fig. 5.) So that we do not need to attach the iron pieces together, we place them vertically and lengthwise, one on top of the other. We again connect the load resistance, build the measuring devices into the circuit, and apply an alternating tension of 50 V to the primary winding.

The difference between $U(1)$ and $U(2)$, as well as between $I(1)$ and $I(2)$, is much greater than in the first demonstration. The efficiency is now only about 40%. With the help of the Hall probe, we can determine very well where the lost energy goes. By moving the Hall probe in the middle between the two arms from the lower to the upper winding, we see that the value of the magnetic tension decreases by approximately 50%. Accordingly, there is a magnetic potential drop within each of the two magnetic conductors. The existence of such a magnetic potential drop is not surprising since, of course, any nonideal magnetic conductor has a nonzero (magnetic) resistance. This loss of energy along the magnetic conductors of a transformer is completely analogous to the energy loss along the (nonsuperconducting) electric conductors of an electric circuit: The dissipation of energy in an electric conductor leads to a drop in the (electric) tension along the conductor.

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