

10 Integrals of motion

Subject:

In theoretical mechanics, integrals of motion play an important role: quantities whose values remain constant in time. A system with n degrees of freedom has $2n - 1$ such integrals. One often calls these quantities conserved quantities:

(1) “A function $f(q, \dot{q}, t)$ is called a conserved quantity or integral of motion, if

$$\frac{df(q, \dot{q}, t)}{dt} = 0$$

or $f(q, \dot{q}, t) = \text{constant}$, holds for all trajectories $q_i(t)$, that fulfill the *Lagrange*-equations [1].”

(2) “Apparently, the momentum is a conserved quantity, if its temporal derivative disappears, i.e. if the forces \vec{K}_1 and \vec{K}_2 are equal and opposite during the entire course of the motion, if thus

$$\vec{K}_1 + \vec{K}_2 = 0.” [2]$$

Deficiencies:

In theoretical or analytical mechanics the expression “conserved quantity” has a different meaning from that in other fields of physics.

In general, i.e. if we refrain from analytical mechanics, one uses the designations “conserved” or “not conserved” in order to characterize a substance-like physical quantity. (A quantity is substance-like if a density and a current can be attributed to it.) Some substance-like quantities are conserved, like energy, momentum and electrical charge, and others are not, for instance entropy. Conservation or non-conservation, respectively, is a universal property of a quantity. It is not the characteristic of a certain function, a certain system or a certain process. It also makes no sense to speak of the conservation or non-conservation of a non-substance-like quantity. Temperature, for instance, is neither conserved nor non-conserved.

In theoretical mechanics, on the contrary, the word “conserved quantity” stands for “integral of the motion” (see our first quotation). An integral of motion is not necessarily substance-like and is often not intuitive. An example is the *Runge-Lenz* vector. The *Runge-Lenz* vector is time-independent for the *Kepler* problem. According to the usage of theoretical mechanics it is a conserved quantity of the *Kepler* problem. However, the *Runge-Lenz* vector is not a substance-like quantity, because no density and no current density can be defined for it. In addition, it is not always time-independent, but only in the *Kepler* problem.

According to the practice of theoretical mechanics the quantities energy, momentum and angular momentum are sometimes conserved and sometimes not (see our second quotation).

Origin:

Theoretical mechanics is one of the most elegant physical theories. It also plays an important role as a basis of other theories: It requires only few modifications to become quantum theory. This perfection may be due to the fact that it was completed quite independently of other fields of physics. Thereby it has developed its own vocabulary. Among other things the designation “conservation” is used in a different sense than elsewhere. This cannot always be noticed because in some cases the meaning overlaps with that in the other areas of physics. This use may also be the cause of a somewhat unfortunate formulation of the true and universal conservation of a quantity. Instead of characterizing a conserved quantity by saying that it cannot be produced or destroyed, it is said that the value of the quantity is constant in a closed system.

Disposal:

One distinguishes between the concepts “integral of motion” and “conserved quantity”, as it is done, for instance, in Landau-Lifshitz [3]: “Among them [the integrals of motion] are some whose constancy has a deeper cause, that is related to the basic properties of time and space – its homogeneity and its isotropy. All these so-called conserved quantities have in common the important property of being additive.”

[1] *F. Kuypers*: Klassische Mechanik. Physik-Verlag, Weinheim, 1983, S. 38.

[2] *W. Macke*: Mechanik der Teilchen. Akademische Verlagsgesellschaft, Leipzig, 1962, S. 240.

[3] *L. D. Landau and E. M. Lifschitz*: Theoretische Physik kurzgefaßt I. Akademie-Verlag, Berlin, 1973, S. 17.

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