The chain fountain with momentum currents

F. Herrmann, Karlsruhe Institute of Technology

At least since the „chain fountain" phenomenon was published by the *New York Times* it is one of the favorite dinner table topics of physicists around the world [1,2]. I had repeatedly been asked, to give an explanation in terms of momentum currents. Indeed, the chain fountain seems to be predestinated as an example for the demonstration of the concept of momentum flow. I wrote this comment, without looking at the literature that already exists.

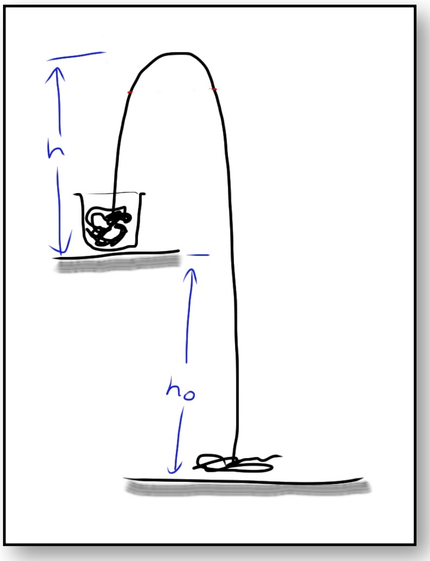


Fig. 1. Notation

We shall see, why the chain ascends, i.e. why it not simply flows over the rim of the upper container. We shall calculate the height of the fountain (*h* in Fig. 1) under the condition that there is no energy dissipation, except where the chain hurts the ground. We shall also define a necessary condition for the dissipation freeness of the starting process of the chain links.

For this purpose we first consider momentum currents and then energy currents. Finally we discuss how the process of starting or jumping up of the chain links has to be in order to achieve dissipation freeness.

Since we repeatedly have to refer to the elements of which the chain consists, we will already reveal the design of the chain's links, Fig. 2: Each link consists of a stick on which a bead is fixed. The beads carry the entire mass of the chain, whereas the sticks are assumed to be massless and they can bent elastically. The sticks are concatenated by means of articulations. The exact disposition of each bead on its stick is essential for the working of the chain fountain. It will be discussed in section 4.

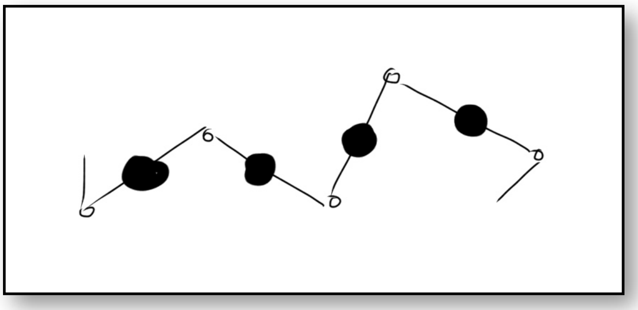


Fig. 2. Each baed is fixed on a stick. The sticks are coupled by articulations.

1. Momentum currents

For our purpose it is sufficient to consider the vertical component of momentum which we call *z* component, and which we define to be positive for a downward movement.

A moving chain that is under tensional stress transports momentum. There are two contributions to the momentum flow: a *convective* current and a *conductive* current.

The convective current is due to the fact that each bead has mass and velocity, thus it has momentum. Since the bead is moving, momentum is moving with the bead.

The conductive momentum current is related to the fact that the sticks are under tensional stress. This component of the momentum flow has nothing to do with the movement of the chain. Its strength is identical with what usually is called a force.

Thus, we can say that the beads carry the convective momentum current and the sticks the conductive current.

We call:

F_c = convective momentum current
F_s = conductive momentum current („S“ from stick)

The convective momentum current

It is equal to the product of velocity v and mass current I_m:
$F_c = v \cdot I_m$
The mass current can be written as
$I_m = \lambda \cdot v$
where λ is the mass of a section of the chain divided by the length of the section.

The convective momentum current thus becomes:
$F_c = v \cdot I_m = \lambda v^2$
The absolute value of the convective current is the same at every position of the chain. Its direction of flow is downwards in both vertical segments of the chain: the left and the right one, Fig. 3. In the right section of the chain positive momentum is moving downwards; in the left section negative momentum moves upwards, which is equivalent to positive momentum moving downwards.

The conductive momentum current

Since the sticks are under tensional stress, the conductive z momentum current goes upwards in both vertical parts of the string, Fig. 3. Moreover, its absolute value is increasing in the upward direction, since on its way up momentum is entering the beads, which comes from the earth, via the gravitational field. The momentum current increase per length l is:
$F_G/l = \lambda \cdot g$.

In the right vertical segment of the chain the conductive current is zero at the lower end, i.e. at position C. At the top, i.e. at position A, it is
$F_G = \lambda \cdot g \cdot (h_0 + h)$

Consider now the left vertical segment of the chain.

At the U-turn at position A no z momentum can be exchanged between the right and the left vertical section of the chain. The z component of both the convective and the conductive contributions to the current are zero at the summit. Thus, no z momentum

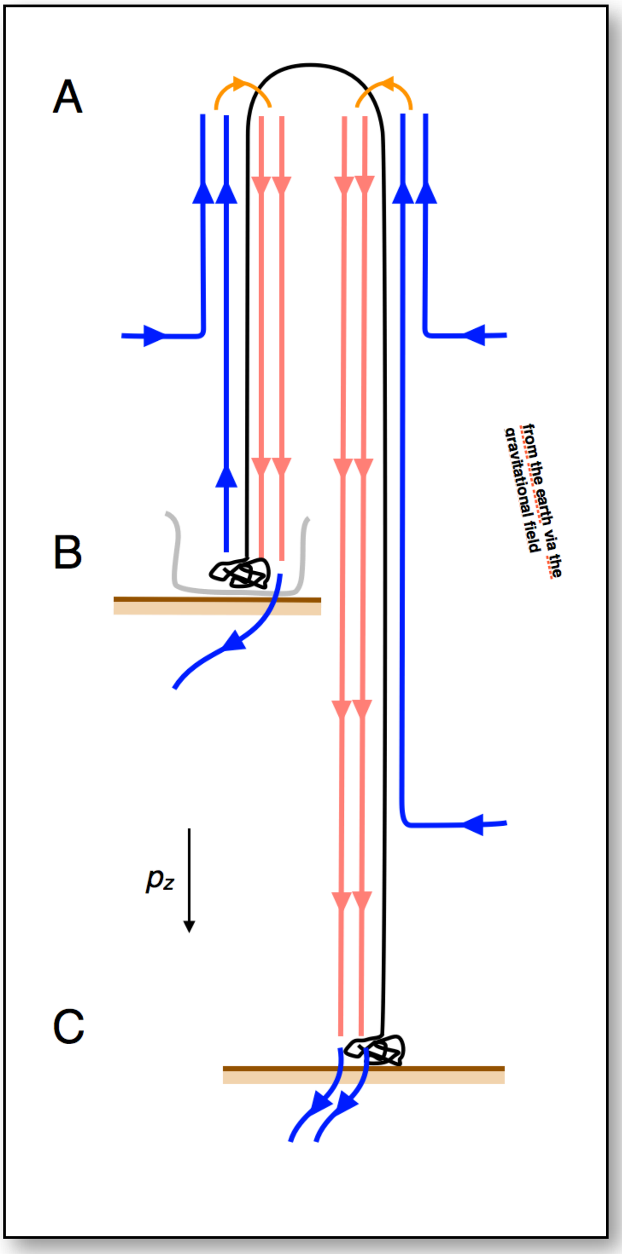


Fig. 3. The convective momentum current (red) is equal in both vertical parts of the chain. It is flowing downwards. The conductive current (blue) is flowing upwards. On its way new momentum enters the chain coming from the earth via the gravitational field.

can cross this point. For this reason, the absolute value of the descending convective flow must be equal to that of the ascending conductive flow. We consider the conductive flow $F_{s,A}$ at the top of the chain (but below the bending):

$F_{s,A} = F_c = \lambda v^2$

At the bottom, i.e. position B, the conductive current is smaller by $\lambda \cdot g \cdot h$. We thus have

$F_{s,B} = F_{s,A} - \lambda \cdot g \cdot h = \lambda v^2 - \lambda \cdot g \cdot h.$

The momentum current λv^2 that arrives at position B by the convective flow must go somewhere. However, the momentum, that flows upwards conductively is not sufficient. We have to get rid of λv^2, but only $\lambda v^2 - \lambda \cdot g \cdot h$ flows upwards. There is an excess of $\lambda \cdot g \cdot h$. If there would be no other outflow of momentum this term would be zero, and h would be zero too; there would be no fountain. An excess of momentum could only flow away into the container of the chain, and thus into the earth (E). The fact that certain chains produce a fountain shows that such an outflow can exist. We still do not know why and how this momentum flow F_E leaves the chain and enters the earth, but we know that it must be:

$$F_E = \lambda \cdot g \cdot h \,. \tag{1}$$

One more word about how the ascending momentum transforms into descending momentum at the top position A. This process does not happen at an exactly defined height, but it takes place in the section where the chain bends from vertical to horizontal.

2. Energy currents

The energy current in the chain can be decomposed in the same manner as the momentum current.

To the convective momentum current a current of kinetic energy is associated:

$$P_{kin} = \frac{\text{kinetic energy}}{\text{time}} = \frac{\text{kinetic energy}}{\text{length}} \cdot v = \frac{\lambda}{2} v^2 \cdot v = \frac{\lambda}{2} v^3$$

Just like the convective momentum current, this component of the energy current has the same absolute value at every position of the chain. In the left segment of the chain it is flowing upwards, i.e. in the opposite direction of the momentum current, whereas in the right segment it flows downwards (like the momentum current), Fig. 4.

In addition, energy is flowing with the conductive momentum current through the sticks. It is related to the conductive momentum current by:

$P_S = v \cdot F_S$

Similarly to the conductive momentum current it increases in the upwards direction, since there is an energy exchange with the gravitational field: In the right segment energy enters the chain, in the left one it leaves the chain. The rate of energy transfer of a section of the chain of length Δz to or from the gravitational field is:

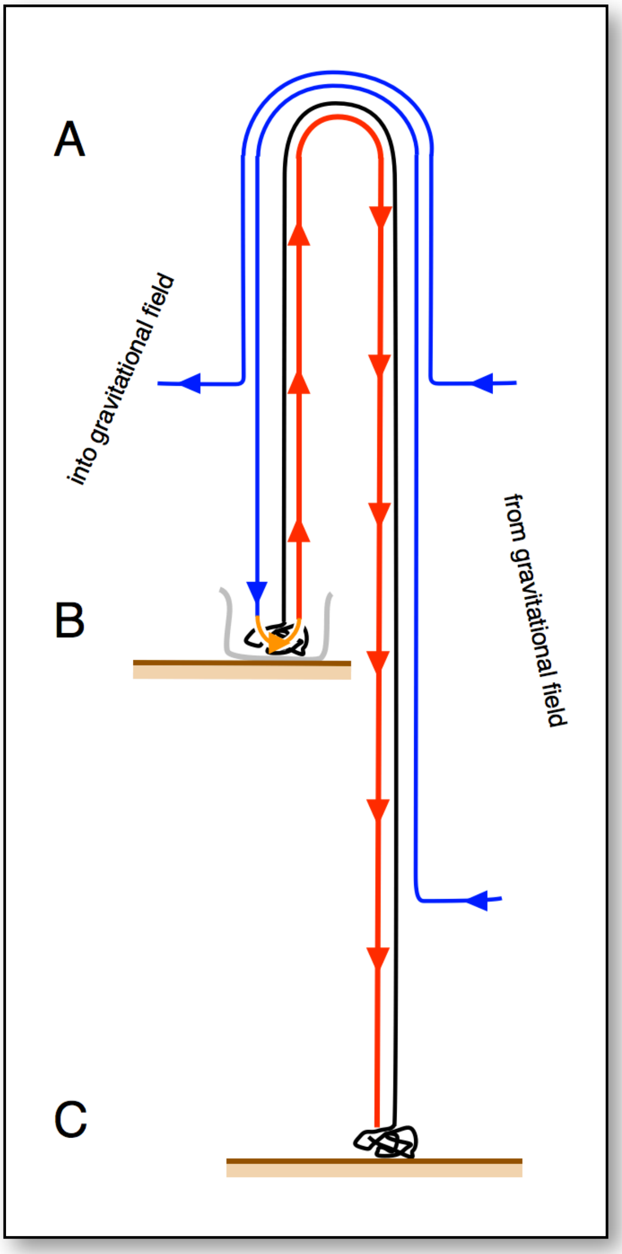
$P_G = \lambda \cdot g \cdot \Delta z \cdot v \quad .$

We consider the upper part of the arrangement, which has the length $\Delta z = h$. At the right side an energy flow of

$P_{G,\,above} = \lambda g h v$

enters the chain. At the left side the same current leaves the chain. We are not interested in this contribution to the total current that only traverses our system. What we are interested in is that energy, which enters the lower part of the right hand segment of the chain. It amounts to:

$P_{G,\,below} = \lambda g h_0 v$



It is flowing upwards, turns left together with the chain and goes down again. It is available for the acceleration of the beads in the container at position B.

3. Dissipation

The acceleration of the beads can be considered a collision process. The moving chain that is catching the next bead from the container acts like a body whose mass is much greater than that of the bead which is to be accelerated.

According to the design of the chain links, the collision can be more or less elastic, from completely inelastic to (at least in theory) completely elastic.

If it is completely elastic, the whole incoming energy flow $\lambda g h_0 v$ goes into the acceleration of the beads:

$$\lambda g h_0 v = \frac{\lambda}{2} v^3$$

from which we get:

$$v^2 = 2 g h_0$$

Fig. 4. Energy comes from the gravitational field and enters the right vertical section of the chain. In the right section of the chain it flows upwards, in the left section it flows down, whereby a part of it leaves the chain and returns to the gravitational field. The remainder can be used to accelerate the beads. It then flows as kinetic energy with the beads and is eventually dissipated where the beads hurt the ground at position C.

If the collision is completely inelastic, the dissipated energy is equal to the kinetic energy of the light body. This result is known from the inelastic collision of a body of great mass against one with a small mass. Thus, in this case only half of the incoming energy goes into the acceleration:

$$\frac{\lambda}{2} g h_0 v = \frac{\lambda}{2} v^3 ,$$

from which we get:

$$v^2 = g h_0 .$$

Thus, in general we have:

$$g h_0 \le v^2 \le 2 g h_0$$

Here we have neglected any other energy loss or dissipation.

4. Realization of the various types of collisions

In the following we consider a single chain link resting horizontally on the ground. At a certain instant of time it will receive an upwards „push“ by the neighboring link which had been accelerated an instant earlier. We can distinguish several cases, according to the position of the bead on its stick.

First imagine the bead in figure 5 to be at the right end of the stick, Fig. 5a: It will be swept along, whereby the velocity it acquires is that of the chain. We know this behavior from the inelastic collision of a massive body against a body of small mass. Half of the transmitted energy is dissipated in the process. But how can it be that the process is inelastic even though the chain's components are elastic? Even if the initial collision is partly elastic, half of the energy must be damped away, possibly by oscillations that decay rapidly. At the end it has to move at the velocity of the chain and that means half of the energy has gone.

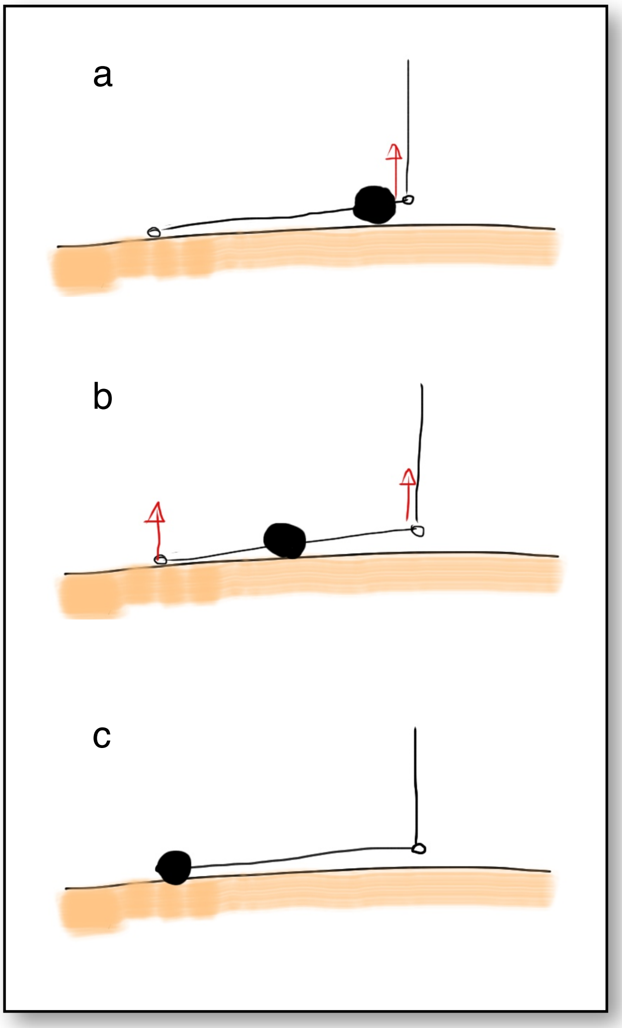


Fig. 5. Only if the bead sits in the middle of the stick, the collision is elastic.

If we place the bead somewhat away from the articulation, its velocity immediately after the collision will be smaller and correspondingly there will be less dissipation. If is placed exactly at the middle of the stick, Fig. 5b, its velocity after the collision is exactly that of the moving chain, and no energy has to be dissipated. When placing the bead even farther to the left, its velocity after the collision will be smaller than that of the chain and the collision again becomes inelastic. It is approaching the position next to an articulation, which corresponds to an inelastic collision.

Now the interesting point: the role of angular momentum. If the bead sits at the middle of the stick, the collision (the „push“ by the moving chain) tries to rotate the stick. Although its moment of inertia is small (we have a small bead on a long stick), it cannot rotate. The rotation is hindered by the earth. The same amount of momentum that leaves the right end of the link and goes upwards in the chain, is also going into the earth via the left end of the link. Thereby the bead gets the corresponding amount of „negative momentum“. In the process the two amounts of angular momentum that enter the stick compensate each other.

We finally understand the origin of the excess momentum flow F_E, (see equation (1)) which determines the height of the fountain according to:

$$F_E = \lambda \cdot g \cdot h$$

It is seen, that half of the momentum that is delivered by the beads goes upwards: the conductive momentum current through the chain; the other half goes into the container or into the earth, each current being equal to $\lambda v^2/2$. We thus have:

$$\lambda gh = \frac{\lambda}{2}v^2$$

or

$$h = \frac{v^2}{2g}$$

Since in the dissipationfree case we have

$$v^2 = 2gh_0$$

the height of the fountain would be:

$h_{max} = h_0$.

In reality there is dissipation, and not only that of the collision process which we have discussed. So we can give intervals for the fountain's height and velocity:

$$0 \le h \le h_0$$

and

$$gh_0 \le v^2 \le 2gh_0 .$$

[1] http://www.nytimes.com/2014/03/04/science/the-chain-fountain-explained.html?_r=0

[2] http://www.youtube.com/watch?annotation_id=annotation_625777&feature=iv&src_vid=_dQJBBklpQQ&v=6ukMId5fIi0