Simple explanation of a well-known collision experiment

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A well-known collision experiment can be carried out with an arrangement of several identical elastic balls each suspended by two threads and in contact with one another: a certain number of the balls is displaced from its equilibrium position and then released, so as to collide with the remaining balls at rest. After the collision, the same number of balls moves away to the other side as had initially been displaced. It is shown that, contrary to common belief, the conservation laws of energy and momentum alone are not sufficient to explain this behavior. Indeed, a further condition must be satisfied by the system of balls; namely, it must be capable of dispersion-free energy propagation.

I. INTRODUCTION

In order to confirm the conservation laws of energy and momentum, the following well-known collision experiment is often carried out in the classroom: several identical elastic balls are suspended from a horizontal frame and in contact with one another in such a way that their movement is restricted to one plane (Fig. 1). If some of the balls are displaced together from their equilibrium position to one side and then released, allowing them to collide with the other balls remaining at rest, it is observed that the same number of balls swings away to the opposite side after the collision as was originally displaced. Furthermore, the same number of balls remains at rest after the collision as were at rest before the collision.

No calculation is required to realize that this behavior agrees with the laws of conservation of energy and momentum. In textbooks describing this experiment, it is sometimes implied,[1,2] sometimes even explicitly stated,[3,4] that the experimental observation is a *necessary* consequence of the conservation laws of energy and momentum. However, it is easy to convince oneself that the conservation laws of energy and momentum are *not sufficient* to explain the observed behavior of the balls.

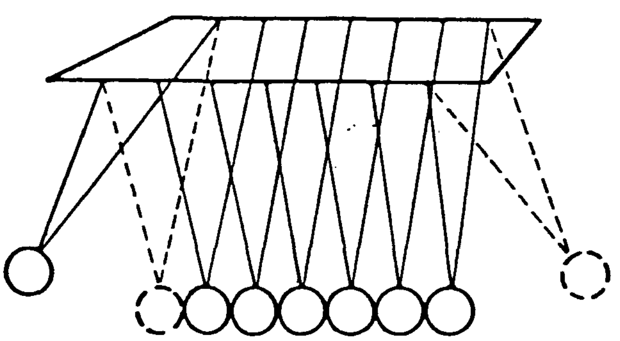

Fig. 1. Same number of balls move away after the collision as was initially displaced and the same number of balls remain at rest.

Let the "state" of the system be defined by the values of the velocities of all the balls. The conservation of energy and momentum during the collision yields two equations governing the final state, if the initial state is known. The final state is determined if the velocities of all the balls after the collision are specified. In other words, the number of equations involving the velocity needed to determine the final state is equal to the number of balls. Therefore, if there are more than two balls, the two global conservation laws are not sufficient to determine the final state.

By way of example, consider an arrangement of three balls each of mass m (Fig. 2). Ball 1 is displaced to the left and then released. Its velocity v_0 immediately before the impact is assumed to be known. With balls 2 and 3 at rest, the initial state is then well defined. The total kinetic energy E_k and the total momentum P are thus

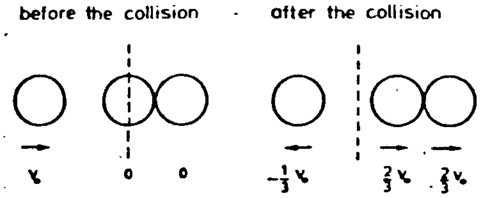

Fig. 2. Initial and final state of a fictitious collision. Though energy and momentum are conserved in the collision the final state shown here is not realized.

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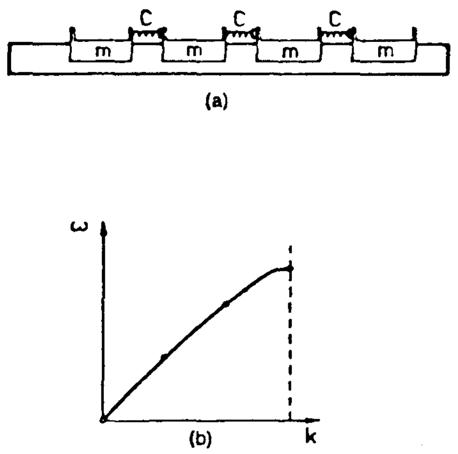

Fig. 3. (a) Arrangement of four gliders coupled by springs has four eigenfrequencies. (b) These frequencies ω_i versus the corresponding wave numbers k_i are not situated on a straight line. The glider arrangement thus exhibits dispersion.

$$E_k = (1/2)mv_0^2,$$
$$P = mv_0.$$

Imagine now the following final state: ball 1 moves to the left with $v = (-1/3)v_0$, balls 2 and 3 move to the right, both with the same speed $v = (2/3)v_0$. It is easy to confirm that the values of the kinetic energy and the momentum of this hypothetical final state are the same as those of the initial state:

$$E_k = (1/2)m[(1/3)v_0]^2 + 2(1/2)m[(2/3)v_0]^2 = (1/2)mv_0^2,$$
$$P = m[(-1/3)v_0] + 2m(2/3)v_0 = mv_0.$$

Thus energy and momentum would be conserved. Nevertheless, the actual experiment always evidences another outcome.

The main point of this paper is to answer the question of why the arrangement of balls always separates in the observed way, leaving the same number of balls at rest after the collision as were at rest before. Since the behavior of the balls is extremely simple, one would expect, that this question has a simple answer.

In Sec. II such an answer is developed by investigating an air-track model of the ball arrangement, which lends itself to detailed analyses. Furthermore, a simple expression for the transit time of the perturbation through the arrangement is derived in Sec. III. In Sec. IV, the relationship between the air-track model and the original physical arrangement of balls is discussed. Finally, in Sec. V, a simple way of intuitively understanding the propagation of the perturbation in the ball arrangement is presented.

II. AIR-TRACK MODEL OF THE BALL ARRANGEMENT

Begin by considering a simplified version of the ball arrangement whereby the N balls were replaced by N gliders on an air track. All gliders have the same mass m and to each of them a spring bumper is attached with a spring of force constant C thus enabling each glider to collide elastically with its neighbor. In contrast to the original ball experiment, elasticity and inertia are now spatially separated.

To carry out a collision experiment, n gliders are moved together as a group with the velocity v_0 against the $N-n$ gliders remaining at rest and in contact with one another. For purposes of illustration we assume the n gliders to be impinging from the left. One might reasonably expect the outcome of this experiment to be the same as that of the original ball experiment: after the collision n gliders "should" move to the right and $N-n$ gliders "should" remain immobile.

Surprisingly this does not happen. After the collision all gliders are seen to move in a seemingly unordered manner. No simple pattern can be recognized in the final state. Obviously, the air-glider system as here proposed is unsuitable to serve as a model of the original ball arrangement.

To see why this glider system does not exhibit the expected behavior and to understand how the system should be modified to correctly model the actual ball arrangement, it is necessary to analyze the above proposed glider model in more detail.

From the instant of first contact of the incoming gliders with those at rest until the instant of separation of the arrangement into two parts, the system can be considered as a "linear chain," i.e., as a linear arrangement of bodies of equal mass coupled by identical springs [Fig. 3(a)]. Such arrangements are typically described in solid-state textbooks for the quantitative treatment of lattice vibrations.[5]

Any vibrational state of the chain can be represented by a linear combination of its normal modes, each of which is characterized by a corresponding eigenfrequency and wave number. When the system vibrates in an eigenmode, all of its components execute a harmonic oscillation with the frequency characteristic of that mode. The number of modes equals the number of degrees of freedom of the system. In our one-dimensional case it is equal to the number of gliders N.

The values of the frequencies ω_i and of the wave numbers k_i of the four modes of the chain shown in Fig. 3(a) are given in Table I. Here a is the distance between the midpoints of two neighboring gliders. Figure 3(b) displays the dispersion relation, i.e., ω as a function of k.

If the masses of the bodies and the force constants of the springs had not been all the same, there would still be only four eigenmodes, but the shape of the dispersion curve would be different from that of Fig. 3(b).

A system of linear oscillation is completely determined by its dispersion relation. Thus the outcome of these collision experiments, which themselves only involve oscillators used in their linear domain, must likewise be completely determined by the dispersion relation of the arrangement. Each system will behave differently in accordance with its own dispersion curve. Now, it is tempting, but logically not at all necessary, to suppose that a system with the simplest dispersion curve, i.e., a straight line, should exhibit the simplest behavior in a collision experiment, i.e., the kind of

Table I. Frequencies ω_i and wave numbers k_i of the four modes of the chain of gliders on an air track shown in Fig. 3(a).

$\omega_1 = 0$	$k_1 = 0$
$\omega_2 = (2-2^{1/2})^{1/2}(C/m)^{1/2}$	$k_2 = (1/3)\pi/a$
$\omega_3 = 2^{1/2}(C/m)^{1/2}$	$k_3 = (2/3)\pi/a$
$\omega_4 = (2+2^{1/2})^{1/2}(C/m)^{1/2}$	$k_4 = \pi/a$

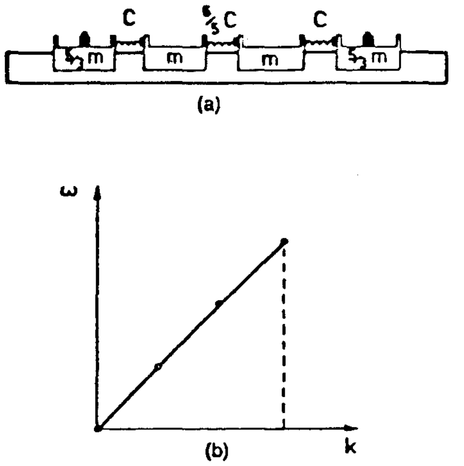

Fig. 4. (a) By modifying masses and force constants of the glider system a dispersion-free arrangement is obtained. (b) The ω vs k relation is now linear.

behavior found with the elastic balls of Fig. 1. To find out if that is true, one can compute the values of the masses of the four gliders, and the force constants of the three springs between them that give rise to a linear dispersion relation. The analysis[6] is somewhat lengthy but straightforward and yields the values shown in Fig. 4(a). The corresponding values of ω_i and k_i are given in Table II, the dispersion curve is shown in Fig. 4(b).

When collision experiments are carried out with the glider chain correspondingly modified to display a linear dispersion curve, our hypothesis is confirmed: The system behaves like the ball arrangement of Fig. 1. The number of gliders scattering to the right is equal to the number of gliders incoming from the left and the number of gliders at rest after the collision is equal to that before. Analogous calculations[6] have been carried out with a five-glider arrangement. The values of glider masses and force constants are indicated in Fig. 5. The corresponding experiments give the same results as those with four gliders.

III. PROPAGATION TIME OF THE DISTURBANCE THROUGH THE SYSTEM

The dispersion-free glider arrangement of Fig. 4(a) exhibits the same collision behavior as the ball arrangement of Fig. 1. The model system has the advantage over the actual ball arrangement in that it lends itself not only to observation, but also to the thorough mathematical treatment of the collision.

As we have mentioned earlier between the instant t_0 of first contact of the incident gliders with the gliders at rest until the instant t_1 of separation the system can be treated as a "linear chain."

We first compute the position as a function of time $x_{ij}(t)$ of every glider i in each of the four eigenstates j, as measured from the equilibrium position of each glider i.

Any general vibrational state of the system is a linear combination of the four eigenstates j with weighing coefficients a_j. The x–t relationship for the ith glider can thus

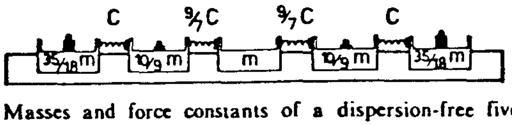

Fig. 5. Masses and force constants of a dispersion-free five-glider system.

Table II. Frequencies ω_i and wave numbers k_i of the four modes of the chain of gliders on an air track shown in Fig. 4(a).

$\omega_1 = 0$	$k_1 = 0$
$\omega_2 = (2/5)^{1/2}(C/m)^{1/2}$	$k_2 = (1/3)\pi/a$
$\omega_3 = 2(2/5)^{1/2}(C/m)^{1/2}$	$k_3 = (2/3)\pi/a$
$\omega_4 = 3(2/5)^{1/2}(C/m)^{1/2}$	$k_4 = \pi/a$

be represented by

$$x_i(t) = \sum_{j=1}^{4} a_j x_{ij}(t). \qquad (1)$$

The coefficients a_j can be calculated if (i) the position and (ii) the velocities of all the gliders are known at any instant t. In the present case these values are known for the instant t_0 of first contact:

(i) All gliders are in their position of equilibrium, i.e., $x_i(t_0) = 0$.

(ii) The velocities of the incident gliders are $v = v_0$; the velocities of the gliders at rest are $v = 0$.

The coefficients a_j have been calculated[6] for (a) one incident glider, three gliders at rest; (b) two incident gliders, two gliders at rest; and (c) three incident gliders, one glider at rest. Case (c) becomes identical with case (b) if the system of reference is fixed on the incoming gliders instead in the laboratory.

When the coefficients a_j are known the further behavior of the four gliders can be predicted by means of Eq. (1). The first instant t_1 after t_0 at which time at least one of the springs shows a dilatational strain is the instant of separation.

The following results are obtained:

(1) The locations of impact and separation are situated symmetrically with respect to the middle of the arrangement with all gliders (including the incident gliders) in contact.

(2) The propagation time of the disturbance $T = t_1 - t_0$ depends on the total number of gliders in the system, but not on the number of incident gliders.

(3) T equals the ratio of the length L of the arrangement with all gliders in contact and the "sound velocity" c in the system:

$$T = L/c.$$

c is equal to the slope of the (linear) dispersion curve.

IV. BALL ARRANGEMENT AS A DISPERSION-FREE SYSTEM

We can summarize the result of our collision experiments on the air track: The glider model exhibits the same, familiar behavior of the chain of balls, when the glider ar-

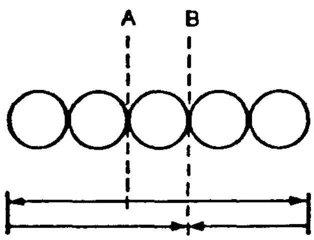

Fig. 6. At point A a disturbance builds up and propagates through the system in opposite directions (upper arrows). After the reflection of both components at the ends of the system, they meet again (lower arrows) at point B. From A to B they traverse the length of the system just once.

rangement is modified to be dispersion-free. Only in the case of a dispersion-free system is it possible to transfer the total energy and momentum of the incoming gliders to the same number of gliders at the other end of the chain. In a non-dispersion-free system energy and momentum are distributed throughout the entire arrangement.

Thus by analogy to our glider model, we conclude that the chain of balls of Fig. 1 is a dispersion-free arrangement.

The perturbation generated by the impact of the incoming balls propagates through the chain of balls without changing its shape and therefore transfers the energy and momentum of the incoming balls to the same number of balls at the other end of the chain.

V. SIMPLE PICTURE OF THE COLLISION PROCESS

The experimental observations of the dispersion-free arrangement and the results presented in Sec. III can be understood in terms of a simple picture.

Imagine a film of a collision experiment with an arrangement of five balls to be shown in slow motion. Balls 1 and 2 are impinging from the left upon balls 3–5, which are at rest (Fig. 6). During the collision process, a disturbance is developed with time at the point of impact A. Starting at A, the disturbance propagates in both directions (upper arrows in Fig. 6). Both components of the disturbance propagate throughout the system without dispersion, are reflected at the ends, run back (lower arrows), and meet at some point B, located such that A and B lie symmetrically about the middle of the entire five ball system.

The entire disturbance that builds up over a finite time period at the point of impact A and that leads to two waves propagating in opposite directions from A along the arrangement of balls, results in the reunition of these waves over an identical finite time period at B without distortion of the shape (i.e., amplitude versus time) of the initial disturbance.

Therefore, the separation of the system can be considered as the exact time-reversed process of the impact. If the entire collision process had been filmed with a movie camera, which is able to record the deformation of the balls, and the film were to be played backwards, the same collision process would be seen with the exception that the point of first contact would then be at B and the point of separation at A.

In this picture, it is easily seen that the entire disturbance transverses the length of the system just once. Thus the transit time equals the length of the system divided by the propagation speed of the disturbance, independent of the number of incident balls.

VI. CONCLUSIONS

The result of collision experiments with a linear arrangement of elastic balls cannot be predicted solely from the conservation laws of energy and momentum, if the number of such balls is greater than two. The behavior that is observed, is a consequence of the dispersion-free character of the system. The disturbance always traverses a distance equal to the length of the system independent of the point of origin of said disturbance. The transit time is thus independent of the number of the balls initially displaced.

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[2] R. Pohl, *Mechanik, Akustik und Wärmelehre* (Springer-Verlag, Berlin, 1969), p. 52.

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[5] J. M. Ziman, *Principles of the theory of solids* (Cambridge University, Cambridge, 1964), p. 30.

[6] The authors of this article will provide details of these calculations upon request.